Midterm (Take Home)

M551 – Abstract Algebra

October 27th, 2014

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* about this exam and you can only look at *our* book, class notes and solutions to *our* HW problems *posted by me* or done by yourself. No other reference, including the Internet.

Due date: noon on Monday (10/27). If you cannot bring it to class or to me, a scanned/typed copy by e-mail would be OK.

1. Let G be a finite group, $P \in \mathrm{Syl}_p(G)$ and $N \stackrel{\text{def}}{=} \mathrm{N}_G(P)$. Prove that if $H \leq G$ with $N \leq H$, then $\mathrm{N}_G(H) = H$.

[**Hint:** Let $M \stackrel{\text{def}}{=} \mathrm{N}_G(H)$. Prove that $\mathrm{Syl}_p(H) = \mathrm{Syl}_p(M)$ and so $n_p(H) = n_p(M)$. Use this to deduce that H and M have the same order.]

2. Let G act on a set S. Define, for $g \in G$,

$$\chi(g) \stackrel{\text{def}}{=} |\{a \in S \ : \ g \in \mathrm{stab}_G(a)\}| \, .$$

[Remember that $\mathrm{stab}_G(a) = \{g \in G \ : \ g \cdot a = a\}$.] Prove that the number of orbits of this action [remember, the orbits form a partition of S, so this is the number of elements in this partition], say n, is given by:

$$n = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

Hint for 2 on the back!

[**Hint:** Let for $a \in S$ and $g \in G$, let

$$\Phi(a, g) = \begin{cases} 1, & \text{if } g \in \text{stab}_G(a); \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,

$$\sum_{a \in S, g \in G} \Phi(a, g) = \sum_{g \in G} \left(\sum_{a \in S} \Phi(a, g) \right) = \sum_{a \in S} \left(\sum_{g \in G} \Phi(a, g) \right).$$

What are the inner parenthesis in the latter two sums equal to?

Also, if $G \cdot a_1, \ldots, G \cdot a_n$ are *all* the orbits [and so n is the number of orbits, as above], for a fixed $i \in \{1, \ldots, n\}$, for how many $a \in S$ do we have that $G \cdot a = G \cdot a_i$?]