

Prelim Exam – Spring 2012 – Problem II.2

December 1, 2014

Let R be a PID and J denote the intersection of all maximal ideals of R . If $a^2 - a \in J$ for all $a \in R$, then show the R has only finitely many maximal ideals.

Proof. If R is a field, then (0) is the only maximal ideal, so we may assume R is not a field.

Let M be a maximal ideal. Since R is not a field, we have that $M \neq 0$. Then, since R is a PID, $M = (m)$, where m is a prime/irreducible element of R [since maximal ideals are prime].

Also, $J = (x)$, for some $x \in R$, and clearly $(x) = J \subseteq M = (m)$, i.e., $m \mid x$. If $x \neq 0$, since R is a UFD [since it is a PID], we have that x has only finitely many non-associate prime/irreducible divisors, which means only finitely many choices for m , up to associates, and so finitely many choices for M .

So, assume that $J = 0$, i.e., $x = 0$. Then, for all $a \in R$, by assumption $a^2 - a \in J = 0$, i.e., $a(a - 1) = a^2 - a = 0$. Since R is a domain, this means that $a = 0$ or $a = 1$. Hence, since this is true for all $a \in R$, we must have $R = \mathbb{Z}/2\mathbb{Z}$, a field, which is a contradiction. \square