Midterm 2 – Solutions

Definition: Let G be a finite Abelian group. Let $I(G)$ be the vector of *invariant factors* of G, i.e., $I(G) = (d_1, d_2, \ldots, d_k)$ if

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z},$$

where $d_{i+1} \mid d_i$ for $i \in \{1, \ldots, (k-1)\}$.

Let also $E(I)$ be the vector of *elementary divisors* [with the order from class, as explained below], i.e., $E(I) = (q_1, q_2, \ldots, q_l)$ if

$$G \cong \mathbb{Z}/q_1\mathbb{Z} \oplus \mathbb{Z}/q_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_l\mathbb{Z},$$

where q_i is a power of a prime [not necessarily distinct], say $q_i = p_i^{r_i}$, p_i prime, and if $i < j$ then either $p_i < p_j$ or we have both $p_i = p_j$ and $r_i \leq r_j$.
[Feel free to talk to me if this definition is not clear.]

1) [20 points] Let

$$G_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/5^2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7^2\mathbb{Z}$$

$$G_2 = \mathbb{Z}/(2^3 \cdot 3^2 \cdot 5 \cdot 7^2)\mathbb{Z} \oplus \mathbb{Z}/(3^2 \cdot 5 \cdot 7)\mathbb{Z}$$

$$G_3 = \mathbb{Z}/2^2\mathbb{Z} \oplus \mathbb{Z}/(3 \cdot 5^2 \cdot 7)\mathbb{Z} \oplus \mathbb{Z}/(2 \cdot 3^2)\mathbb{Z} \oplus \mathbb{Z}/(3 \cdot 7^2)\mathbb{Z}$$

(a) Compute $E(G_i)$, $I(G_i)$ for $i = 1, 2, 3$.

Solution. We have:

$$I(G_1) = (2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2, 2 \cdot 3 \cdot 7, 3) \qquad E(G_1) = (2, 2^2, 3, 3, 3^2, 5^2 7, 7^2)$$

$$I(G_2) = (2^3 \cdot 3^2 \cdot 5 \cdot 7^2, 3^2 \cdot 5 \cdot 7) \qquad E(G_2) = (2^3, 3^2, 3^2, 5, 5, 7, 7^2)$$

$$I(G_3) = (2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2, 2 \cdot 3 \cdot 7, 3) \qquad E(G_3) = (2, 2^2, 3, 3, 3^2, 5^2 7, 7^2)$$

□

(b) Find all pairs $i, j \in \{1, 2, 3\}$ with $i < j$ such that $G_i \cong G_j$. [Justify!]

Solution. Since $G_i \cong G_j$ iff $I(G_i) = I(G_j)$ [or $E(G_i) = E(G_j)$], from (a) we have that $G_1 \cong G_3$ and no other pair. □

(c) If $P_7 \in \mathrm{Syl}_7(G_2)$, then what is P_7 isomorphic to?

Solution. Since [from $E(G_2)$ in (a)]:

$$G_2 \cong \mathbb{Z}/2^3\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7^2\mathbb{Z},$$

we have

$$P_7 \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7^2\mathbb{Z}.$$

□

2) The *exponent* of a group G is the smallest positive integer k [if exists] such that $g^k = 1$ for all $g \in G$. Prove that if G is a finite Abelian group, then its exponent [exists and] is d_1, where d_1 is the first entry of $I(G)$. [So, with the *additive* notation, we need to prove that d_1 is the smallest positive integer such that $d_1 \cdot g = 0$ for all $g \in G$.]

Proof. We have that

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z},$$

where $d_{i+1} \mid d_i$ for $i \in \{1, \ldots, (k-1)\}$. We may assume that we actually have an equality [as order of elements is preserved by isomorphisms]. Let $g = (g_1, g_2, \ldots, g_k)$. Since $g_i \in \mathbb{Z}/d_i\mathbb{Z}$, we have that $d_i \cdot g_i = 0$. Since d_1 is a multiple of d_i [as $d_i \mid d_1$], we also have $d_1 \cdot g_i = 0$, and hence $d_1 \cdot g = (d_1 \cdot g_1, \ldots, d_1 \cdot g_k) = (0, \ldots, 0) = 0$. Hence, the exponent exists and if we denote it by e we have that $e \leq d_1$.

Now, on the other hand, we have that $e \cdot (1, 0, 0, \ldots, 0) = 0$, and then $e = 0$ in $\mathbb{Z}/d_1\mathbb{Z}$ and hence $d_1 \mid e$. Thus, [since $e \leq d_1$] we have that $e = d_1$. □

3) Let G be a finite Abelian group such that there is $H \leq G$ such that $H \neq \{0\}$ [using additive notation] and if $K \leq G$ then $H \leq K$.

(a) Prove that if such H exists, then G is cyclic of order power of a prime.

Proof. Consider the decomposition of G into its *elementary* factors. Suppose that there is more than one factor. Let $\mathbb{Z}/p^r\mathbb{Z}$ be the first and K be the direct sum of the remaining factors. [Since we are assuming there is more than one, we have that $K \neq \{0\}$.] Then, $G = \mathbb{Z}/p^r\mathbb{Z} \oplus K$. We have that $H_1 \stackrel{\text{def}}{=} \mathbb{Z}/p^r\mathbb{Z} \oplus \{0\}$ and $H_2 \stackrel{\text{def}}{=} \{0\} \oplus K$ are non-trivial subgroups of G, and hence $H \leq H_1, H_2$, i.e., $H \leq H_1 \cap H_2 = \{0\}$, which is a contradiction.

Thus, G has only one elementary factor, and hence it is cyclic of order power of a prime. □

(b) Give an example of a *non-Abelian* group of finite order for which such H does exist. [Just give me G and H. No need to justify.]

Solution. We have that $H = \{1, -1\}$ is $G = Q_8$ works. □

4) Let G be a group of order $3 \cdot 7 \cdot 11$.

(a) Prove that G has normal subgroup, say H, of order 77.

Proof. From Theorem 7.3.10(c) we get that $n_7 = n_{11} = 1$. So, if P_7 and P_{11} are Sylow 7 and 11-subgroups of G respectively, we have $P_7, P_{11} \lhd G$. Thus, we have that $H \stackrel{\text{def}}{=} P_7 \cdot P_{11} \lhd G$. We just need to now prove that $|H| = 77$: we have, by the Third Isomorphism Theorem, that $|H| = (|P_7| \cdot |P_{11}|)/\,|P_7 \cap P_{11}|$. But since P_7 and P_{11} have relatively prime orders, we must have that $P_7 \cap P_{11} = \{1\}$, and hence $|H| = 77$. □

(b) Prove that if G does not have exactly 7 subgroups of order 3, then G is cyclic.

Proof. From Theorem 7.3.10(c) we get $n_3 \in \{1, 7\}$. So, since $n_3 \neq 7$, then $n_3 = 1$ and, as on (a), we have that if $P_3 \in \text{Syl}_3(G)$, then $P_3 \triangleleft G$.

So, with the group H from (a), we have that $P_3 \cdot H \triangleleft G$. By the Third Isomorphism Theorem, similarly as done above, we get that $|P_3 \cdot H| = |P_3| \cdot |H| = 3 \cdot 77 = |G|$. Hence, $P_3 \cdot H = G$. Since also $P_3 \cap H = \{1\}$ and $P_3, H \triangleleft G$, we get that $G = P_3 \times H$.

Now P_3 is cyclic [prime order] and so is H [by Problem 7.3.5(c)]. But their orders are relatively prime, and hence $G = P_3 \times H \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/77\mathbb{Z}$ is cyclic. [One can see that, for instance, from the fact that $I(G) = (3 \cdot 7 \cdot 11)$. □

5) Let p and q be distinct primes and let G be a group of order $|G| = p^2 q$. Prove that G has either a normal Sylow p-subgroup or a normal Sylow q-subgroup.
[**Hint:** This is about *sizes and counting*. To derive a contradiction, assume there is neither. Use the *Sylow Theorems* to get that $q > p$. Then, what are n_p and n_q?]

Proof. By Theorem 7.3.10(c), we have $n_p \in \{1, q\}$ and $n_q \in \{1, p, p^2\}$. Assume that there is no normal Sylow p or q-subgroup. Then, $n_p \neq 1$ [and thus $n_p = q$] and $n_q \neq 1$ [and thus n_q is p or p^2].

Since $n_p = q > 1$, we have that $q \equiv 1 \pmod p$ [by Theorem 7.3.10(c) again], and hence $q > p$. But this means that $p \not\equiv 1 \pmod q$, and so $n_q = p^2$.

Now we count elements. We have $n_q = p^2$ subgroups of order q. Since these have prime order, they don't intersect except at the identity. This gives us $p^2(q-1) = p^2q - p^2$ elements of order q [$q - 1$ for each Sylow q-subgroup].

Now we have at least 2 Sylow p-subgroups of order p^2. One of them gives me p^2 elements, which includes the identity and $p^2 - 1$ elements of order either p or p^2. Hence, these p^2 elements are *not* among the ones we've counted above. This gives a total of $(p^2q - p^2) + p^2 = p^2q = |G|$ elements. But we have a *different* Sylow p-subgroup introducing at least one extra element [which is not in the Sylow p-subgroup we've counted, nor among the elements of order q]. Hence, we would have at least $p^2q + 1$ elements, which is a contradiction.

Thus, either $n_p = 1$ or $n_q = 1$. □