What's Calculus?

Answer: Next semester! (Fundamental Theorem of Calculus, by Newton and Leibniz.)

Virtually all of modern science uses calculus! Physics, engineering, statistics, biology (modeling), etc.

This semester: *Differential* Calculus. (Tangent lines.)
Next semester: *Integral* Calculus. (Areas.)

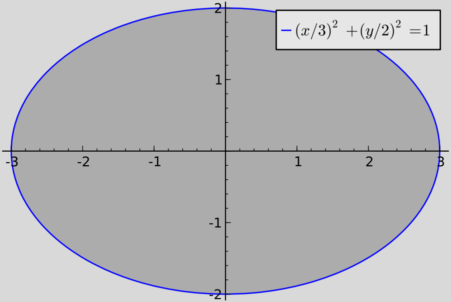
Areas

Computing *areas* is one the most classical problems in mathematics. (The term *geometry* comes from the Greek *"land (or earth) measurement"*.) The idea is to compare the space taken by a plane shape with the space taken by one square of side 1.

- **Area of rectangle:** length of base times length of height.
- **Area of triangle:** half of the length of base times length of height. (From this, we can get areas of polygons.)
- **Area of Circle:** π times the square of the radius. *Why????* How did one find that out?

Other Areas

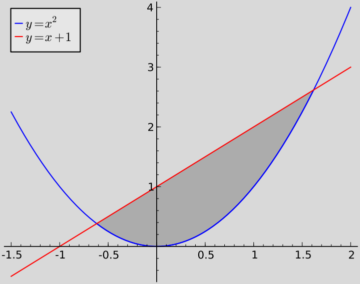
How about the area of an *ellipse*? Say $(x/3)^2 + (y/2)^2 = 1$?

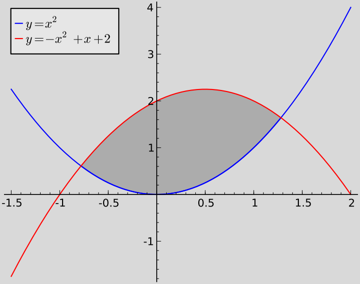


What's its area?

Other Areas

How about the area between a line and a parabola? How about two parabolas?

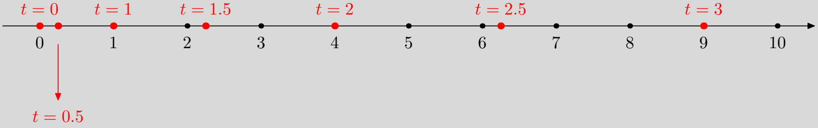




These are hard questions! Answers in Math 142.

Movement

Suppose that you know that a particle in moving along a straight line such that t seconds after we start observing the movement, the position of the particle is t^2 meters from the original position. In other words, the position of the particle is given by the function $s(t) = t^2$.



One can clearly see that the particle is **accelerating**.

Average Speed

Since we know the position at any time, we should be able to find out *everything* about the movements of the particle! (Not only its position at a given time.)

For instance, we can find the **average speed** of the particle in a period. For instance, the average speed between $t = 1$ and $t = 2$:

$$\frac{\Delta s}{\Delta t} = \frac{s(2) - s(1)}{2 - 1} = \frac{4 - 1}{2 - 1} = 3.$$

Between $t = 2$ and $t = 3$:

$$\frac{\Delta s}{\Delta t} = \frac{s(3) - s(2)}{3 - 2} = \frac{9 - 4}{3 - 2} = 5.$$

Instantaneous Speed

But how about the **instantaneous speed** at, say $t = 2$. (What is the speedometer showing if we look at it at $t = 2$?) *Much harder!*
Idea: I might not be able to know the *exact* speed, but I can get a very good idea: find the *average* speed of a *tiny* interval starting at $t = 2$. The smaller the interval is, the less time the particle had to change its speed, so the closest we get to the real speed at $t = 2$!
So, we find the average speed between $t = 2$ and $t = 2 + \Delta t$, for Δt *small*. Here are some computations:

Δt	Aver. Sp.
0.1	4.1
0.01	4.01
0.001	4.001
0.0001	4.0001

So, the speed at $t = 2$ is pretty close to 4.0001. (Is it 4?)

Computing Average Speeds

The computations done for the average speed on the previous slide can be done quite quickly by a computer (or even calculator). But imagine for a second we have to compute *lots* of different average speeds *by hand*!

Here is a smart way of doing it :*find a formula for the average speed!* The average speed between $t = t_0$ and $t = t_0 + \Delta t$ is:

$$\begin{aligned}
\frac{\Delta s}{\Delta t} &= \frac{s(t_0 + \Delta t) - s(t_0)}{(t_0 + \Delta t) - t_0} = \frac{(t_0 + \Delta t)^2 - t_0^2}{\Delta t} \\
&= \frac{(t_0^2 + 2t_o\Delta t + (\Delta t)^2) - t_0^2}{\Delta t} \\
&= \frac{2t_o\Delta t + (\Delta t)^2}{\Delta t} = \frac{2t_o\cancel{\Delta t} + (\Delta t)^{\cancel{2}}}{\cancel{\Delta t}} \\
&= 2t_0 + \Delta t.
\end{aligned}$$

Computing Average Speeds

This makes it easy to compute average speeds and estimate instantaneous speeds:

t_0	Δt	Aver. Sp.
1	2	4
2.5	0.01	5.01
3	0.01	6.01
4	0.01	8.01

In particular, the *instantaneous* speed at $t = 2.5$ is approximately 5.01, the *instantaneous* speed at $t = 3$ is approximately 6.01, the *instantaneous* speed at $t = 4$ is approximately 8.01.

Instantaneous Speed

But how do we find the *exact* instantaneous speed? The idea is that we want $\Delta t = 0$. But this doesn't seem to make sense:

$$\frac{\Delta s}{\Delta t} = \frac{s(t_0 + 0) - s(t_0)}{(t_0 + 0) - t_0} = \frac{s(t_0) - s(t_0)}{t_0 - t_0} = \frac{0}{0}!$$

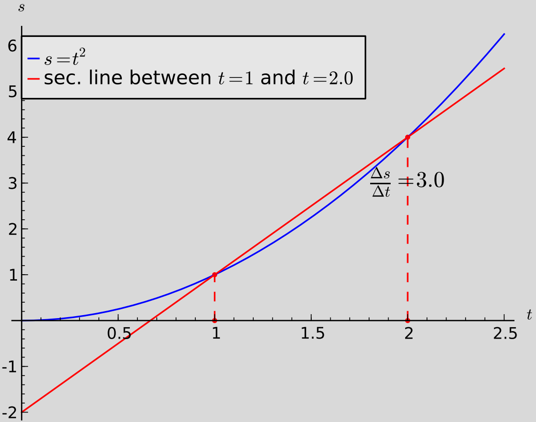
But we cannot divide by 0!

On the other hand, we have a formula for the average speed $\Delta s/\Delta t$: $2t_0 + \Delta t$. So, *here*, we can make $\Delta t = 0$ *without dividing by* 0! Hence, the **instantaneous speed of the particle at** $t = t_0$ **is** $2t_0$.

So, the (instantaneous) speed at $t = 2$ is 4, the (instantaneous) speed at $t = 3$ is 6, the (instantaneous) speed at $t = 4$ is 8, etc.

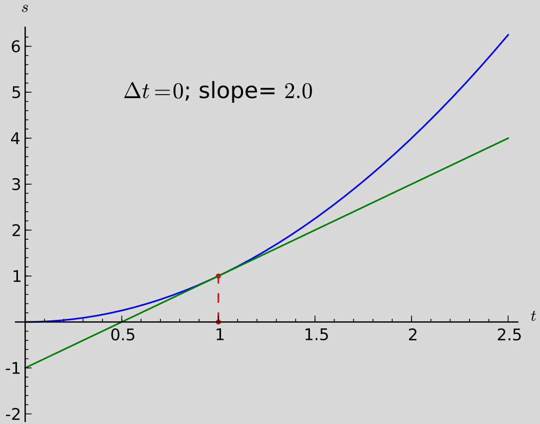
Geometrical Interpretation of Average Speed

Now let's look at the geometry of the average speed. The formula $\frac{\Delta s}{\Delta t}$ is basically a *slope* ($\frac{\Delta y}{\Delta x}$). **The average speed between $t = t_0$ and $t = t_0 + \Delta t$ is the slope of the line secant to the graph of $s(t)$ through $t = t_0$ and $t = t_0 + \Delta t$.**



Geometrical Interpretation of Instantaneous Speed

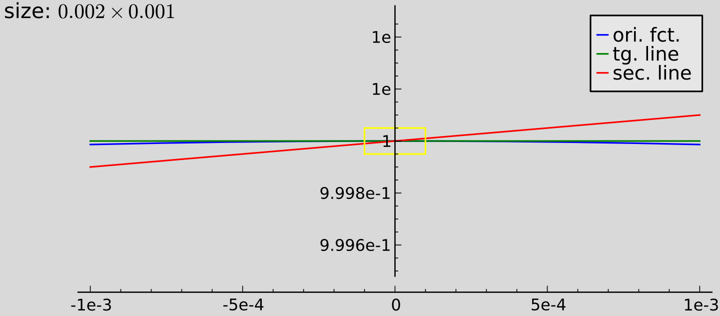
So, what is the geometrical interpretation of the *instantaneous* speed? It is the slope of the **tangent line** at $t = t_0$!



Tangent Line

The **tangent line** is geometrically defined precisely as in the previous pictures: take secant lines and make the second point go approach the first.

Another way to see it: if a curve is smooth (no sharp edge), by *zooming in enough*, it starts to look like a straight line. This straight line is the tangent line! (A line which is not tangent makes an angle!)



Rates of Change

A bit of terminology: the **rate of change** of the position of a particle is its *instantaneous* speed. More precisely, it is how fast the position change when the time changes.

As we've seen, the speed (i.e., rate of change) of the position $s(t)$ at $t = t_0$ is the *slope of the tangent line to the graph* $s = s(t)$ *at* $t = t_0$.

In general, the rate of change of a function $f(x)$ at $x = x_0$ is the *slope* of the tangent line to the graph $y = f(x)$ at $x = x_0$. This tells us how fast is the y value changing at $x = x_0$.

Computations

As we've seen, to compute rates of change (or slopes of the tangent line) of $y = f(x)$ at $x = 0$, we do:

- Consider the ratio: $\dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$;
- simplify so that we don't have a Δx in the denominator;
- replace Δx by 0.

Example

Consider $f(x) = x^2 - x$. What is the slope of the tangent line at $x = 1$?

$$
\begin{aligned}
\frac{f(1+\Delta x) - f(1)}{\Delta x} &= \frac{[(1+\Delta x)^2 - (1+\Delta x)] - (1^2 - 1)}{\Delta x} \\
&= \frac{[(1^2 + 2\Delta x + (\Delta x)^2) - (1+\Delta x)] - 0}{\Delta x} \\
&= \frac{\Delta x + (\Delta x)^2}{\Delta x} \\
&= 1 + \Delta x.
\end{aligned}
$$

Now, we can make $\Delta x = 0$ in the above expression, obtaining the answer: 1.

Problems

There are two problems: the above is not *mathematically precise*! It is a procedure, but does not *define* rate of change/slope of tangent line precisely.

Also, does the procedure always work? Consider the tangent line of $y = sin(x)$ at $x = 0$. How do you simplify

$$\frac{\sin(0 + \Delta x) - \sin(0)}{\Delta x} = \frac{\sin(\Delta x)}{\Delta x}$$

to cancel out the Δx in the denominator (so that we can replace it by 0)? *Hard!*

We need the notion of **limit**!