1) [15 points] Determine if B is a basis of the corresponding vector space V or not, giving a *short* explanation. [**Hint:** None of them actually require computations!]

(a) $B = \{1 + x, 2 + 2x\}$ for $V = P_1$. [P_1 is the vector space of polynomials of degree at most 1.]

Solution. No, as the set is linearly dependent, as the second polynomial is a multiple of the first. □

(b) $B = \{(-1, 2, 3), (0, 1, 4)\}$ for $V = \mathbb{R}^3$.

Solution. No, as the dimension of $\mathbb{R}^3$ is 3, so every basis should have 3 vectors. [The given set does not generate all of $\mathbb{R}^3$, as we need at least three vectors for that.] □

(c) $B = \{(1, 1), (-1, 1)\}$ for $V = \mathbb{R}^2$.

Solution. Yes. Since B has two elements, and 2 is the dimension of $\mathbb{R}^2$, we only need to check that the vectors are linearly independent. Since one vector is not a multiple of the other, the set is linearly independent, and hence B is a basis.

□

2) Change of basis:

(a) [10 points] Let $B = \{(1,1),(0,1)\}$ and $B' = \{(2,1),(1,1)\}$. Give the transition matrix $P_{B \to B'}$.

Solution.

$$\left[\ B' \mid B\ \right] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right].$$

So,

$$P_{B \to B'} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

□

(b) [5 points] Let B and B' be bases of a vector space V, with transition matrix

$$P_{B \to B'} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Then, if $[\mathbf{v}]_B = (2,-1)$, find $[\mathbf{v}]_{B'}$.

Solution. We have

$$[\mathbf{v}]_{B'} = P_{B \to B'} \cdot [\mathbf{v}]_B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

So, $[\mathbf{v}]_{B'} = (1,5)$.

□

3) [15 points] Let $\mathbf{v}_1 = (0, 1, -2, 3)$, $\mathbf{v}_2 = (-2, 1, 1, 1)$, $\mathbf{v}_3 = (1, -1, 1, 0)$. To find if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent or dependent, you need to set up a system, and depending on what happen with the system you can give the answer. Give this system *and* tell how solving it would tell you if the set is linearly independent or dependent [something like, "it is linearly dependent if the system is consistent"].

[**Note:** Again, this is *part* of what you would have to do to find the answer, but I am saving you from actually having to solve the system! If you still don't know what I mean, just determine if the set is linearly dependent or independent.]

Solution. We have that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent if, and only if, the only real numbers such that $k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + k_3 \mathbf{v}_3 = \mathbf{0}$ are $k_1 = k_2 = k_3 = 0$. In other words, if and only if, the unique solution of

$$\begin{bmatrix} 0 & -2 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $k_1 = k_2 = k_3 = 0$, i.e., if the [homogeneous] system above has only one solution. [If it has infinitely many solutions, then it has a non-trivial one, and thus the set would be linearly dependent.] $\square$

4) [15 points] Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $B = \{A_1, A_2, A_3, A_4\}$ is a basis of $M_{2\times 2}$. [Just take my word.] Set up a linear system $A\mathbf{x} = \mathbf{b}$ whose unique solution $\mathbf{x}$ is exactly $[A]_B$ [the coordinates of A with respect to the basis B], where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

[**Note:** Yet again, this is *part* of what you would have to do. If you don't know what I mean, just compute $[A]_B$.]

Solution. We have that $[A]_B = (k_1, k_2, k_3, k_4)$ means that $A = k_1\,A_1 + k_2\,A_2 + k_3\,A_3 + k_4\,A_4$. So,

$$k_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Simplifying the left hand side gives:

$$\begin{bmatrix} k_1 + k_2 + k_3 & k_2 \\ k_3 & k_1 + k_2 + k_3 + k_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Comparing the entries gives:

$$\begin{aligned} k_1 + k_2 + k_3 \qquad &= 1 \\ k_2 \qquad\qquad &= 2 \\ k_3 \qquad &= 3 \\ k_1 + k_2 + k_3 + k_4 &= 4 \end{aligned}$$

The unique solution of this system of this system is then $[A]_B$. $\square$

5) Vector Spaces:

(a) [7 points] Is the set V of vectors (x, y) of $\mathbb{R}^2$ such that $x \geq 0$ a vector space [with the usual addition and scalar multiplication of vectors]? [Show work!]

Solution. No. Although $(0, 0)$ is in V, and summing two elements of V gives another element of V, V is not closed under scalar multiplication: $(1, 1)$ is in V, but $-1 \cdot (1, 1) = (-1, -1)$ is not in V. $\square$

(b) [8 points] The set $\mathbb{R}^2$ with the usual addition but with scalar multiplication given by $k(x, y) = (ky, kx)$ is *not* a vector space. Show all axioms that fail by giving concrete examples. [Use the list in the end.]

Solution. Note that since the addition is the usual one, all axioms that refer only to sum [namely, (1), (2), (3), (4), and half of (0)] are automatically satisfied. Checking the others we see that only (7) and (8) fail: take $\mathbf{u} = (0, 1)$, and $k = l = 1$. Then

$$k(l\mathbf{u}) = 1(1\,(0, 1)) = 1\,(1, 0) = (0, 1),$$

but

$$(kl)\mathbf{u} = (1 \cdot 1)(0, 1) = 1\,(0, 1) = (1, 0).$$

Since these are different, (7) fails. With the same $\mathbf{u}$ we can see that (8) also fails as

$$1\,(0, 1) = (1, 0) \neq (0, 1).$$

$\square$

(c) [10 points] Show that the set V of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 0$ is a vector space [with the usual addition and scalar multiplication of functions]. [Show work!]

Solution. Yes. Since all functions $f : \mathbb{R} \to \mathbb{R}$ is a vector space, we can just check that V is a subspace:

- The function constant equal to zero is clearly in V. [To check if a function is in V we compute its value at $x = 1$. If the value is 0, then it is in V. If it is not, then it is not.]
- If $f(1) = 0$ and $g(1) = 0$, then $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$. So, if $f, g \in V$, then $f + g \in V$.
- If $f(1) = 0$ and $k \in \mathbb{R}$, then $(kf)(1) = k\,f(1) = k\,0 = 0$. So, if $f \in V$, then $kf \in V$.

$\square$

Vector Space Axioms

A non-empty set V with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $\mathbf{u}+\mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$, and $k\mathbf{u} \in V$ for all $\mathbf{u} \in V$ and $k \in \mathbb{R}$;
1. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3. there is $\mathbf{0} \in V$ such that $\mathbf{0}+\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$;
4. given $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$;
5. $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$;
6. $(k+l)\mathbf{u} = k\mathbf{u}+l\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
7. $k(l\mathbf{u}) = (kl)\mathbf{u}$ for all $\mathbf{u} \in V$ and $k, l \in \mathbb{R}$;
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.