Final (Take-Home Part)

M551 – Abstract Algebra

December 3rd, 2007

1. Let R be a commutative ring with identity. Suppose that for each prime ideal P, the localization R_P has no non-zero nilpotent element.

(a) Show that R has no non-zero nilpotent element.

Proof. Let $a \in R - \{0\}$. Let $I \stackrel{\text{def}}{=} \{r \in R \,:\, ra = 0\}$. Then I is an ideal: if $r, s \in I$, then $(r - s)a = ra - sa = 0$. If $x \in R$, then $xra = x \cdot 0 = 0$.

Since $a \neq 0$, we have that $1 \notin I$, and hence $I \neq R$. Thus, there is a maximal ideal M [and hence also prime] such that $I \subseteq M$. But, if $a^m = 0$ [in R], then $(a/1)^m = (a^m)/1 = 0/1$ [in R_M], and hence, since R_M has no non-zero nilpotent elements, we have that $a/1 = 0/1$. So, there exists $x \in R - M$ such that $xa = 0$. But, by definition, such x would have to be in M, giving us a contadiction. Thus, a cannot be nilpotent.

□

(b) Is R necessarily a domain?

Proof. **No!** Let $R \stackrel{\text{def}}{=} \mathbb{Z}/6\mathbb{Z}$. The only proper non-zero ideals of R are $P_1 \stackrel{\text{def}}{=} (2)$ and $P_2 \stackrel{\text{def}}{=} (3)$. [Note that every ideal of R is principal, since $\mathbb{Z}$ is a PID. It is not a PID since it is not a domain, though.] Since it is not a domain, we have that (0) is not prime. By observing containment, we can see that P_1 and P_2 are maximal, and hence prime.

Note that in R_{P_1}, we have that $2/1 = 0/1$, since $3 \notin P_1$ and $3 \cdot 2 = 0$ [in R]. Now, if $(a/b)^n = 0/1$ in R_{P_1}, then there exists $x \notin P_1$ such that $xa^n = 0$. Since $x \notin P_1$, we have that $x = 1, 3, 5$. Since 1 and 5 are units in R, if $a \neq 0$, then we must have $x = 3$. But then, a^n must be in $P_1 = \{0, 2, 4\}$. Since P_1 is prime, we have that $a \in P_1$. So, by our previous remark [i.e., $2/1 = 0/1$], we have $(a/b) = 0/1$. Therefore, R_{P_1} has no non-zero nilpotent elements.

[In fact, we have $R_{P_1} = \{0/1, 1/1\} \cong \mathbb{Z}/2\mathbb{Z}$, since $2/1 = 0$, and $1/1 = 1/3 = 1/5$.]

Note that in R_{P_2}, we have that $3/1 = 0/1$, since $2 \notin P_3$ and $2 \cdot 3 = 0$ [in R]. Now, if $(a/b)^n = 0/1$ in R_{P_2}, then there exists $x \notin P_2$ such that $xa^n = 0$. Since $x \notin P_2$, we have that $x = 1, 2, 4, 5$. Since 1 and 5 are units in R, if $a \neq 0$, then we must have $x = 2$ or $x = 4$. But then, a^n must be in $P_2 = \{0, 3\}$. Since P_2 is prime, we have that $a \in P_2$. So, by our previous remark [i.e., $3/1 = 0/1$], we have $(a/b) = 0/1$. Therefore, R_{P_2} has no non-zero nilpotent elements.

[In fact, we have $R_{P_2} = \{0/1, 1/1, 2/1\} \cong \mathbb{Z}/3\mathbb{Z}$, since $3/1 = 0$, and $1/2 = 2/1$.]

□

2. Let R be a non-Noetherian commutative ring with identity, and $\mathcal{S}$ be the set of ideals which are *not* finitely generated.

(a) Show that $\mathcal{S}$ has a maximal element I. [The ideal I in the next items is this maximal element.]

Proof. We use Zorn's Lemma: let $\mathcal{C}$ be a chain in $\mathcal{S}$. [Note that $\mathcal{S} \neq \varnothing$ since R is non-Noetherian.] Let $I_{\mathcal{C}} \stackrel{\text{def}}{=} \bigcup_{I \in \mathcal{C}} I$. Then, as usual, $I_{\mathcal{C}}$ is an ideal. If $I_{\mathcal{C}}$ is finitely generated, say $I_{\mathcal{C}} = (a_0, \ldots, a_n)$, then there exists $I_i \in \mathcal{C} \subseteq \mathcal{S}$ such that $a_i \in I_i$. Since $\mathcal{C}$ is a chain [i.e., totally ordered], we have that all a_i are in a single I_j, which we can assume, without loss of generality, to be I_n. But then, $I \subseteq I_n \subseteq I$. i.e., $I = I_n$. So, $I \in \mathcal{S}$, which would mean that I is not finitely generated, giving us a contradiction. Thus, $I \in \mathcal{S}$ is an upper bound of $\mathcal{C}$.

□

(b) Suppose that $x \notin I$. Prove that there exists a *finitely generated* ideal $I_0 \subseteq I$, such that $(I_0, x) = (I, x)$. [Don't forget the $I_0 \subseteq I$ part!]

Proof. Since I is maximal in $\mathcal{S}$ and $x \notin I$, we have that $I \subsetneqq (I, x)$, and so $(I, x) \notin \mathcal{S}$, and so it's finitely generated, say $(I, x) = (a_1, \ldots, a_n)$. Since $a_i \in (I, x)$, for each i there exists $b_i \in I$ and $r_i \in R$ such that $a_i = b_i + xr_i$. Let then $I_0 \stackrel{\text{def}}{=} (b_1, \ldots, b_n)$. Clearly $I_0 \subseteq I$, and so $(I_0, x) \subseteq (I, x)$.

Now, given $a + xr \in (I, x)$, since $(I, x) = (a_0, \ldots, a_n)$, we have that there are $s_1, \ldots, s_n \in R$ such that

$$a + rx = s_1 a_1 + \cdots s_n a_n = s_1 b_1 + \cdots + s_n b_n + x(s_1 r_1 + \cdots + s_n r_n) \in (I_0, x).$$

Thus, $(I_0, x) = (I, x)$.

□

(c) Suppose $xy \in I$, but $x, y \notin I$. Prove that $J \stackrel{\text{def}}{=} \{r \in R \,:\, rx \in I\}$ is a finitely generated ideal.

Proof. Let $r, s \in J$. Then, $(r - s)x = rx - sx \in I$, since $rx, sx \in I$, and so $r - s \in J$. Given $t \in R$, we have $trx \in I$, since $(rx) \in I$ and I is an ideal. Thus, J is an ideal.

Now, if $r \in I$, clearly $rx \in I$, and so $I \subseteq J$. But, since $y \notin I$, and $yx = xy \in I$, we have that $I \subsetneqq J$. By the maximality of I in $\mathcal{S}$, we have that J is finitely generated.

□

(d) Prove that I must be prime. [Of course, use (b) and (c). Assume that I is not prime and conclude that it must be finitely generated.]

Proof. Observe that $I \neq R$, since $R = (1)$ and hence not in $\mathcal{S}$ [while $I \in \mathcal{S}$]. Suppose then that $xy \in I$, with $x, y \notin I$. Let J be the ideal from part (c). We claim that $I = (I_0, xJ)$. Indeed, clearly $I_0, xJ \subseteq I$. Now, given $a \in I \subsetneq (I, x) = (I_0, x)$, there are $a_0 \in I_0$ and $r \in R$ such that $a = a_0 + xr$. But then, $xr = a - a_0 \in I$, and hence $r \in J$. So, $a = a_0 + rx \in (I_0, xJ)$, and thus $I = (I_0, xJ)$.

So, since J is finitely generated by (c), if $J = (c_1, \cdots, c_m)$, then $I = (I_0, xJ) = (b_1, \ldots, b_n, xc_1, \ldots, xc_m)$, and I is finitely generated. But this contradicts the fact that $I \in \mathcal{S}$. Therefore, I must be prime.

□

[Note that this proves that if every prime ideal of a commutative ring with 1 is finitely generated, then the ring is Noetherian.]