December 12th, 2007

Math 251

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Fall 2007

Name: ....

Student ID (last 5 digits): XXX-X................

Final

You have two hours to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last five digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 6 questions and 12 printed pages (including this one and a page for scratch work in the end).

No books, notes or calculators are allowed on this exam!

Show all work! Even correct answers without work may result in point deductions. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	10	
2	10	
3	22	
4	24	
5	10	
6	24	
Total	100	

1) [10 points] Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_4, 2x_1 - 4x_2 + 3x_3 + x_4, x_3 + x_4, x_1 - 2x_2 + x_3 + 2x_4).$$

Find all vectors $\mathbf{x} \in \mathbb{R}^4$ [if any] such that $T(\mathbf{x}) = (2, 0, 0, 2)$ and all vectors [if any] such that $T(\mathbf{x}) = (1, 1, 1, 1)$.

2) [10 points] Fill in the blanks in the table below. [No need to justify this one.] Here T_A, as usual, represents the linear transformation associated to the matrix A.

size of A	3×3	3×3	3×3	5×9	9×5	3×1	6×5
rank of A	2						
rank of A^{T}							
dim. of row spc.			1				
dim. of col spc.							
nullity of A							0
nullity of A^{T}				4			
T_A 1-to-1 (y/n)					Y	N	
T_A onto (y/n)		Y					

3) Let $S = \{(1,0,1,2,1), (0,1,1,-1,2), (-1,2,1,-4,3), (2,1,2,-1,1), (0,0,1,4,3)\}$ and let $V = \operatorname{span} S$ [the subspace of $\mathbb{R}^5$ spanned by the set S]. Given that

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & 2 \\ -1 & 2 & 1 & -4 & 3 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} \xrightarrow{\text{red. ech. form}} \begin{bmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -5 & -1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 2 & -1 & -4 & -1 & 4 \\ 1 & 2 & 3 & 1 & 3 \end{bmatrix} \xrightarrow{\text{red. ech. form}} \begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

answer the following.

(a) [5 points] Find a basis of V made of vectors in S.

(b) [5 points] If B is the basis you've found in part (a), express the vectors in S that are not in B as a linear combination of vectors in B.

(c) [6 points] Find a basis for the orthogonal complement $V^{\perp}$ of V.

(d) [6 points] Find a basis of $\mathbb{R}^5$ containing a basis of V you found in part (b).

4) Let P_2 be the vector space of polynomials of degree less than or equal to 2. For $\mathbf{p}$ and $\mathbf{q}$ in P_2, define the inner product by:

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1/2)\mathbf{q}(1/2) + \mathbf{p}(1)\mathbf{q}(1).$$

For example, $\langle x, x^2 + 1 \rangle = 0 \cdot 1 + (1/2) \cdot (5/4) + 1 \cdot 2 = 21/8$.

(a) [6 points] Is $B' = \{2x^2 - 3x + 1, 2x^2 - x, -4x^2 + 4x\}$ an *orthonormal* basis of P_2?

(b) [6 points] Find $(1)_{B'}$, $(x)_{B'}$, and $(x^2)_{B'}$.

(c) [6 points] If $B = \{1, x, x^2\}$ is the standard basis of P_2, find the transition matrix from B to B'.

(d) [6 points] Suppose $(\mathbf{p})_{B'} = (1, 0, -2)$ and $(\mathbf{q})_{B'} = (1, 1, 1)$ and let θ be the angle between them [with respect to the given inner product]. Compute, $\langle \mathbf{p}, \mathbf{q} \rangle$, $\|\mathbf{p}\|$, and $\cos\theta$.

5) [10 points] Let

$$A = \begin{bmatrix} 9 & 5 & -5 \\ -8 & -7 & 8 \\ -8 & -11 & 12 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Then, we have that $P^{-1}AP = D$. [You don't have to check it! Just take my word for it.] Find a matrix B such that $B^2 = A$. [So, in some sense, $B = A^{1/2}$, where A is diagonalizable.]

6) Let

$$A = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}.$$

(a) [6 points] Give the characteristic equation of A and show that the eigenvalues are 0 and -6.

(b) [6 points] Find the eigenspaces for each eigenvalue. [Since I've given you the eigenvalues, i.e., 0 and -6, you can do this part even if you didn't do part (a).]

(c) [6 points] Find an *orthogonal* matrix P such that $P^{-1}AP$ is diagonal.

(d) [6 points] Give the diagonal matrix $P^{-1}AP$.

Scratch: