1) Answer all giving short explanations.

(a) Let V be a vector space and $\mathbf{v} \in V$. When is $\{\mathbf{v}\}$ linearly independent? [No need to explain this one.]

Solution. Remember, a single vector is linearly independent if, and only if, this vector is non-zero. [Since a linear combination of the single vector is $k\mathbf{v}$. If $k\mathbf{v} = \mathbf{0}$, then either $k = 0$ or $\mathbf{v} = \mathbf{0}$. So, it's linearly independent if, and only if, $\mathbf{v} \neq \mathbf{0}$.]

□

(b) Is the set $\{(1, 2, 3, 4, 5), (-2, -4, -6, -8, -9)\}$ linearly independent [in $\mathbb{R}^5$]?

Solution. Yes, since one is not a multiple of the other.

□

(c) Is the set $\{(-5, \sqrt{2}), (\pi, \mathrm{e}), (\ln(3), 1/2)\}$ linearly independent [in $\mathbb{R}^2$]?

Solution. No, since $\dim \mathbb{R}^2 = 2$ and we have 3 vectors. [More vectors than the dimension *always* gives us linearly dependent sets.]

□

(d) Does the set $\{1 + x + x^3, -2 + x^2, 1 + x - x^2 + x^3\}$ span all of P_3 [i.e., all polynomials of degree less than or equal to 3]?

Solution. No, since $\dim P_3 = 4$ and we only have 3 vectors. [Less vectors than the dimension of the space cannot span the space.]

□

2) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator for which

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Find the matrix $[T]$ associated to the linear transformation T.

Solution. We have

$$[T] = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix}$$

But $T(2\mathbf{e}_2) = (2, -2, 6)$, and since T is linear, we have that $T(2\mathbf{e}_2) = 2T(\mathbf{e}_2)$. So, $T(\mathbf{e}_2) = (1, -1, 3)$.

We have that $T(\mathbf{e}_1 + \mathbf{e}_3) = (3, 2, 1)$. But since T is linear, we have that $T(\mathbf{e}_1 + \mathbf{e}_3) = T(\mathbf{e}_1) + T(\mathbf{e}_3)$. So, $T(\mathbf{e}_3) = (3, 2, 1) - (2, -1, 0) = (1, 3, 1)$.

Thus,

$$[T] = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

□

(b) Is T one-to-one? Is it onto? [Don't forget to justify!!]

Solution. We have that $\det[T] = -22 \neq 0$, we have that T is both onto and one-to-one.

□

3) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ and W be the *range* of T. In other words, the elements of W are of the form $T(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^m$. Prove that W is a vector space.

Solution. Since $W \subseteq \mathbb{R}^n$ [with the same addition and scalar multiplication], we just need to show it is a *subspace* of $\mathbb{R}^n$, which is a lot simpler.

[Note that W is not empty, since, for instance, $T(\mathbf{0}) = \mathbf{0} \in W$.]

Two elements of W are of the form $T(\mathbf{v})$ and $T(\mathbf{w})$ where $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$. Then, since T is linear, $T(\mathbf{v})+T(\mathbf{w}) = T(\mathbf{v}+\mathbf{w})$. Since $\mathbf{v}+\mathbf{w} \in \mathbb{R}^m$, we have that $T(\mathbf{v})+T(\mathbf{w}) = T(\mathbf{v}+\mathbf{w}) \in W$. [So it's closed under addition.]

Now, if $k \in \mathbb{R}$, then, since T is linear, $kT(\mathbf{v}) = T(k\mathbf{v})$. Since $k\mathbf{v} \in \mathbb{R}^n$, we have that $kT(\mathbf{v}) = T(k\mathbf{v}) \in W$. [So it's closed under scalar multiplication.]

So, since W in a non-empty subset of $\mathbb{R}^m$ which is closed under addition and scalar multiplication, we have that W is a subspace of $\mathbb{R}^m$ and hence it is itself a vector space.

Here is another solution. Let $[T]$ be the matrix associated to T. Then the range of T is the set of vectors $T(\mathbf{x}) = [T] \cdot \mathbf{x}$ such that $\mathbf{x} \in \mathbb{R}^m$. But, if $[T] = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_m]$ [i.e., the $\mathbf{c}_i$'s are the columns of $[T]$], then

$$[T] \cdot \mathbf{x} = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_m] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\mathbf{c}_1 + \cdots x_m\mathbf{c}_m,$$

where $x_1, \ldots, x_m \in \mathbb{R}$. Hence, the range of T is $\operatorname{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$, i.e., the *column sapce* of $[T]$. Therefore, it's a vector space [in fact, a subspace of $\mathbb{R}^n$].

$\square$

4) Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 \\ 1 & 0 & -1 & 0 & 2 \\ 2 & 2 & 2 & 0 & 4 \end{bmatrix}$$

(a) Find bases for the nullspace, column space, and row space of A, with the requirement that the basis for the *column* space of A is composed of columns of A. [There is no requirement for the row and null spaces.]

Solution. Putting A in reduced echelon form, we get:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, the basis for the row space of A is

$$S_{\text{row}} = \{(1, 0, -1, 0, 2), (0, 1, 2, 0, 0), (0, 0, 0, 1, -2)\}.$$

Since the first, second, and fourth columns have the leading ones, we get that a basis for the column space A [made of columns of A] is:

$$S_{\text{col}} = \{(0, 0, 1, 2), (1, 0, 0, 2), (0, 2, 0, 0)\}.$$

For the nullspace, note that the general solution of $A\mathbf{x} = \mathbf{b}$ is [from the echelon form]:

$$\mathbf{x} = \begin{bmatrix} t - 2s \\ -2t \\ t \\ 2s \\ s \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

So, the basis for the nullspace of A is

$$S_{\text{null}} = \{(1, -2, 1, 0, 0), (-2, 0, 0, 2, 1)\}.$$

□

(b) Let S be the basis for the column space that you've found in (a). Then, for each column $\mathbf{c}_i$ of A, find $(\mathbf{c}_i)_S$ [i.e., write the coordinate vector of this column with respect to the basis S].

Solution. Let $\mathbf{c}'_i$ denote the columns of the echelon form. Then, we can easily see that

$$\mathbf{c}'_3 = -\mathbf{c}'_1 + 2\mathbf{c}'_2 + 0\mathbf{c}'_4 \qquad \text{and} \qquad \mathbf{c}'_5 = 2\mathbf{c}'_1 + 0\mathbf{c}'_2 - 2\mathbf{c}'_4.$$

So, we have:

$$\mathbf{c}_3 = -\mathbf{c}_1 + 2\mathbf{c}_2 + 0\mathbf{c}_4 \qquad \text{and} \qquad \mathbf{c}_5 = 2\mathbf{c}_1 + 0\mathbf{c}_2 - 2\mathbf{c}_4.$$

Since, $S = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$, we have $(\mathbf{c}_3)_S = (1, 2, 0)$ and $(\mathbf{c}_5) = (2, 0, -2)$. Also, clearly $(\mathbf{c}_1)_s = (1, 0, 0)$, $(\mathbf{c}_2)_S = (0, 1, 0)$, and $(\mathbf{c}_4)_S = (0, 0, 1)$.

□