1) Let $\mathbf{u} = (1, -1, 2)$ and $\mathbf{v} = (2, 0, 1)$.

(a) Is the angle between $\mathbf{u}$ and $\mathbf{v}$ greater than $\pi/2$, smaller than $\pi/2$ or equal to $\pi/2$? [Remember, the angle between two vectors is always in the interval $[0, \pi]$.]

Solution:

We have $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 2 + (-1) \cdot 0 + 2 \cdot 1 = 4 > 0$. So the smaller than $\pi/2$. [The cosine being positive, means that the angle is in the first quadrant.]

(b) Compute $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

Solution:

We have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{4}{2^2 + 0^2 + 1^2} (2, 0, 1) = \left(\frac{8}{5}, 0, \frac{4}{5} \right)$$

(c) Compute the *length* of the orthogonal component of $\mathbf{u}$ with respect to the direction of $\mathbf{v}$.

Solution:

We can just use Pythagoras [draw a picture!]: if $\mathbf{w}$ is the orthogonal component, then $\|\mathbf{u}\|^2 = \| \operatorname{proj}_{\mathbf{v}} \mathbf{u}\|^2 + \|\mathbf{w}\|^2$. Hence,

$$\|\mathbf{w}\| = \sqrt{6 - \frac{80}{25}} = \sqrt{\frac{70}{25}} = \frac{\sqrt{70}}{5}.$$

[Or, you could remember that $\mathbf{w} = (\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u})$, and compute its length.]

2) If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, then compute [justifying or showing work!]:

(a) $\begin{vmatrix} a/7 & b/7 & c/7 \\ -2d & -2e & -2f \\ g & h & i \end{vmatrix} =$

Solution:

$= \frac{1}{7}\begin{vmatrix} a & b & c \\ -2d & -2e & -2f \\ g & h & i \end{vmatrix} = \frac{-2}{7}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \frac{-2}{7} \cdot 3 = -\frac{6}{7}.$ [Factor 1/7 from the first row and -2 from the second.].

(b) $\begin{vmatrix} f & d & e \\ c & a & b \\ i & g & h \end{vmatrix} =$

Solution:

$= -\begin{vmatrix} c & a & b \\ f & d & e \\ i & g & h \end{vmatrix} = \begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3.$ [Switch first and second row, then switch first and second column, then switch second and third column.]

(c) $\begin{vmatrix} a-2d & b-2e & c-2f \\ g & h & i \\ d+g & e+h & f+i \end{vmatrix} =$

Solution:

$= \begin{vmatrix} a-2d & b-2e & c-2f \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a-2d & b-2e & c-2f \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3$

[Subtract the second row from the third, switch second and third rows, add twice the second row to the first.]

3) Let $A = \begin{bmatrix} 3 & 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 \\ -3 & 5 & 1 & 1 & 3 \\ 2 & 0 & 1 & 0 & 2 \\ 1 & -1 & 2 & 0 & 4 \end{bmatrix}$.

(a) How many solutions does the *homogeneous* system $A\mathbf{x} = \mathbf{0}$ have? [You do **not** have to find the solutions!!! Just tell me how many and justify.]

Solution:

$$\begin{vmatrix} 3 & 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 \\ -3 & 5 & 1 & 1 & 3 \\ 2 & 0 & 1 & 0 & 2 \\ 1 & -1 & 2 & 0 & 4 \end{vmatrix} = - \begin{vmatrix} 3 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & -1 & 2 & 4 \end{vmatrix} \qquad \text{[using the 4th col.]}$$

$$= - \left(2 \begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix} \right) \qquad \text{[using the 2nd row]}$$

$$= - \left(0 + \begin{vmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix} \right) \qquad \text{[a col. is a mult. of another]}$$

$$= -(0 + 1 - 2 - (0 - 3 + 4)) = 2$$

Thus, $\det A \neq 0$, and since the system is homogeneous [i.e., $\mathbf{b} = \mathbf{0}$], we have *exactly one solution* [namely $\mathbf{x} = \mathbf{0}$].

(b) Is A^{T} invertible? If so, compute $\det((A^{\mathrm{T}})^{-3})$, if not, justify.

Solution:

We have $\det A^{\mathrm{T}} = \det A = 2 \neq 0$. Thus A^{T} is invertible.

So,

$$\det((A^{\mathrm{T}})^{-3}) = (\det A^{\mathrm{T}})^{-3} = (\det A)^{-3} = 2^{-3} = \frac{1}{8}.$$

4) Let $A = \begin{bmatrix} -4 & 5 & 2 \\ 1 & -2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$. Solve, if possible, the two systems:

$$A\mathbf{x} = \mathbf{b} \qquad \text{and} \qquad A\mathbf{x} = \mathbf{c}.$$

Solution:
Solve simultaneously:

$$\left[\begin{array}{ccc|c|c} -4 & 5 & 2 & 1 & 2 \\ 1 & -2 & -1 & 0 & 1 \\ -2 & 1 & 0 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & -2 & -1 & 0 & 1 \\ -4 & 5 & 2 & 1 & 2 \\ -2 & 1 & 0 & -1 & 4 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c|c} 1 & -2 & -1 & 0 & 1 \\ 0 & -3 & -2 & 1 & 6 \\ 0 & -3 & -2 & -1 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & -2 & -1 & 0 & 1 \\ 0 & -3 & -2 & 1 & 6 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & -2 & -1 & 0 & 1 \\ 0 & 1 & 2/3 & -1/3 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right].$$

So, $A\mathbf{x} = \mathbf{b}$ has no solutions. As for $A\mathbf{x} = \mathbf{c}$, we have

$$x_2 = -2 - \frac{2}{3}x_3, \qquad x_1 = 1 + 2x_2 + x_3 = -3 - \frac{1}{3}x_3.$$

Thus, the solutions are

$$\mathbf{x} = \begin{bmatrix} -3 - t \\ -2 - 2t \\ 3t \end{bmatrix}, \qquad \text{for all } t \in \mathbb{R}.$$

[Or,

$$\mathbf{x} = \begin{bmatrix} -3 - t/3 \\ -2 - 2t/3 \\ t \end{bmatrix}, \qquad \text{for all } t \in \mathbb{R}.]$$

5) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 & -4 \\ -2 & 5 & 7 \\ -1 & 2 & 4 \end{bmatrix}$. Compute $A \cdot (B^{\mathrm{T}})^{-1}$.

Solution:
Let's compute $(B^{\mathrm{T}})^{-1}$:

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ -4 & 5 & 2 & 0 & 1 & 0 \\ -4 & 7 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -3 & -2 & 4 & 1 & 0 \\ 0 & -1 & 0 & 4 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & -3 & -2 & 4 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & -2 & -8 & 1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & -7 & 0 & -2 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 4 & -1/2 & 3/2 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & -1/2 & -1/2 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 4 & -1/2 & 3/2 \end{array}\right]$$

So,

$$A \cdot (B^{\mathrm{T}})^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & -1/2 & -1/2 \\ -4 & 0 & -1 \\ 4 & -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} -2 & -\frac{3}{2} & \frac{1}{2} \\ -4 & 0 & -1 \end{bmatrix}.$$