Groups of Order 18

Math 455 – Fall 2006

Question: Classify all groups of order 18.

Solution. Let $|G| = 18 = 2 \cdot 3^2$ and s_p denote the number of Sylow p-subgroups. Then, by the *Sylow Third Theorem*, $s_3 \mid 2$ [and so $s_3 \in \{1, 2\}$] and $s_3 \equiv 1 \pmod 3$. Thus $s_3 = 1$ and we have only one subgroup of order 9, say H.

Since $|H| = 9$, by Corollary 6.1.14, $H \cong C_9$ or $H \cong C_3 \times C_3$.

Now, $s_2 \in \{1, 3, 9\}$ [since $s_2 \mid 9$]. We split the problem in cases.

Case 1: Assume that $s_2 = 1$. Then, if K is the [only] subgroup of order 2, we have that $K \lhd G$ [since $s_2 = 1$], $H \cap K = \{1\}$ [since the groups have relatively prime orders], and $|H|\,|K| = |G|$. So, by Proposition 2.8.6, we have that $G \cong H \times K$. Therefore,

$$\boxed{G \cong C_3 \times C_3 \times C_2 \cong C_3 \times C_6}$$

[if $H \cong C_3 \times C_3$], or

$$\boxed{G \cong C_9 \times C_2 \cong C_{18}}$$

[if $H \cong C_9$].

Case 2: Assume that $s_2 = 9$. This means that there are 9 elements of order 2 [one in each Sylow 2-subgroup]. Since H already has 9 elements, this means that every element not in H has order 2.

Let $y \in G - H$. [Hence y has order 2.] Then $G = H \mathbin{\dot\cup} Hy$ [since $[G : H] = 2$, or using Proposition 2.8.6 again]. If $h \in H$, then hy has order 2 [since it is not in H], i.e.,

$$\begin{aligned} (hy)^2 = 1 &\Rightarrow hyhy = 1 \\ &\Rightarrow yhy = h^{-1} \\ &\Rightarrow yh = h^{-1}y. \end{aligned}$$

(i) If $H \cong C_9$ and $H = \langle x \rangle$, then we have that

$$G = H \,\dot{\cup}\, Hy = \{1, x, x^2, \dots, x^8, y, xy, x^2, \dots, x^8y\},$$

the order of x is 9, the order of y is 2 and $yx^i = x^{-i}y$. Thus,

$$\boxed{G \cong D_{18}.}$$

[D_{18} is the *dihedral group of order* 18, as we've seen in class. It is completely characterized by the properties given above.]

(ii) If $H \cong C_3 \times C_3$, we can write

$$H = \{1, x_1, x_1^2, x_2, x_2^2, x_1x_2, x_1^2x_2, x_1x_2^2, x_1^2x_2^2\}$$

where x_1 and x_2 have order 3 and commute with each other. Hence,

$$\begin{aligned} G = H \,\dot{\cup}\, Hy = \{&1, x_1, x_1^2, x_2, x_2^2, x_1x_2, x_1^2x_2, x_1x_2^2, x_1^2x_2^2, \\ &y, x_1y, x_1^2y, x_2y, x_2^2y, x_1x_2y, x_1^2x_2y, x_1x_2^2y, x_1^2x_2^2y\}, \end{aligned}$$

and for any $h \in H$, $yh = h^{-1}y$. We have not encountered this group before, but we can check that these properties indeed give us a group: the properties allows us to make a multiplication table and check all the requirements. [Note for example that, since x_1 and x_2 commute with each other, we have:

$$(x_1^i x_2^j y)^2 = x_1^i x_2^j y x_1^i x_2^j y = x_1^i x_2^j (x_1^i x_2^j)^{-1} y^2 = x_1^i x_2^j x_1^{-i} x_2^{-j} = 1.$$

So, every element not in H indeed has order 2, as we knew it should be the case.] This group is in fact the *semi-direct product of* $C_3 \times C_3$ *with* C_2, but we haven't seen those.

Case 3: Assume, finally, that $s_2 = 3$. We have in this case only 3 elements of order 2. The elements of H can have order 1 [the identity], 3, or 9 [if any]. Therefore, all the elements left [i.e., not of order 2 and not in H] must have order 6, since their orders has to divide $|G| = 18$, and cannot be equal to 1 [because $1 \in H$], 2 [because we're excluding those], 3 [these must be in H], 9 [these, if exist, must also be in H], or 18 [since if we have an element of order 18, G would be cyclic, and hence all subgroups would be normal, and we would have to have $s_2 = 1$, not 3]. Hence, we have 9 elements [of order 1, 3 or 9] in H, 3 elements of order 2, and 6 elements of order 6.

Let y be an element of order 2. So, $y \in G - H$, and as before, $G = H \,\dot{\cup}\, Hy$.

(i) If $H = C_9$, let x be a generator. Let's find what are the other two elements of order 2 [besides y]. If $x^i y$ [which is how every element not in H can be represented] has order 2, then, as before,

$$\begin{aligned}(x^i y)^2 = 1 \Rightarrow x^i y x^i y = 1 \\ \Rightarrow y x^i y = x^{-i} \\ \Rightarrow y x^i = x^{-i} y.\end{aligned}$$

But then,

$$\begin{aligned}yx^{ki} &= yx^i x^{(k-1)i} = x^{-i} y x^{(k-1)i} \\ &= x^{-i} y x^i x^{(k-2)i} = x^{-2i} y x^{(k-2)i} \\ &\vdots \\ &= x^{-(k-1)i} y x^i = x^{-ki} y,\end{aligned}$$

and thus

$$(x^{ki}y)^2 = x^{ki} y x^{ki} y = x^{ki} x^{-ki} y y = 1.$$

Hence, if $x^i y$ has order 2, so does $x^{2i} y$, $x^{3i} y$, etc. Since we have only 3 elements of order 2, they have to be $\{y, x^3 y, x^6 y\}$.

We can conclude that xy has order 6 [since it's not in H and doesn't have order 2]. But then, $(xy)^2$ has order 3. But the only elements of order 3 [which must be in H] are x^3 and x^6, and so,

$$(xy)^3 = (xy)^2 xy = x^3 xy = x^4 y$$

or

$$(xy)^3 = (xy)^2 xy = x^6 xy = x^7 y.$$

But $(xy)^3$ must have order 2 [since xy has order 6], so $(xy)^3 \in \{y, x^3 y, x^6 y\}$, giving us a contradiction, which means that if $s_2 = 3$, then $H \not\cong C_9$.

(ii) So, assume that $H = C_3 \times C_3$. Then, there is some element of Hy besides y that also has order 2. [Remember that y has order 2.] Let $x \in H - \{1\}$ be an element such that xy has order 2. Again, this means that $yx = x^{-1} y$, and we can easily check that $x^2 y$ also has order 2. [Note that since $x \in H - \{1\}$, it must have order 3.] So, the subset

$$S \stackrel{\text{def}}{=} \{1, x, x^2, y, xy, xy^2\}$$

satisfy: x has order 3, y has order 2, and $yx = x^2 y$. So, S is in fact a *subgroup* and is isomorphic to S_3.

We will now prove that $S \lhd G$: let $g \in G$ and $a \in S$. [We need to show that $gag^{-1} \in S$.] If a has order 2 [i.e., $a \in \{y, xy, x^2 y\}$], then gag^{-1} also has order 2. But since S contain all 3 elements of order 2, we must have that $gag^{-1} \in S$. If a does not have order 2, then $a \in H \cap S$ [more precisely $a \in \{1, x, x^2\}$]. But since $G = H \cap yH$, we can write $g = y^i h$, where $i \in \{0, 1\}$ and $h \in H$. But then, $gag^{-1} = y^i hah^{-1} y^i$ [since $y^i = y^{-i}$]. Since $h, a \in H$ and H is commutative, we have that $hah^{-1} = a$. Thus, $gag^{-1} = y^i a y^i$. Since $a, y \in S$, we have that $gag^{-1} = y^i a y^i \in S$. Therefore, $S \lhd G$.

Now, let $\mathcal{S}$ be the set of subgroups of order 3 in G. Then $|\mathcal{S}| = 4$ [i.e., the 4 subgroups of order 3 of $H \cong C_3 \times C_3$]. Then, $\langle y \rangle = \{1, y\}$ acts on $\mathcal{S}$ by conjugation [since conjugation does not change the order]. So, the orbits must have order 1 or 2 [since it must divide $|\langle y \rangle| = 2$]. Also, since $S \lhd G$, the orbit of $\{1, x, x^2\}$ has only one element [namely, itself]. So,

$$4 = |\mathcal{S}| = 1 + \sum_{O \text{ orbit}} |O|$$

So, there must be another orbit of order 1. [If not, the right hand side would be odd!] Let K be this subgroup, and so $yKy = K$.

We will now show that $K \lhd G$. Let $g \in G$. Then, as before, $g = y^i h$, with $i \in \{0, 1\}$ and $h \in H$. Since $K < H$ and H is Abelian, we have that $hKh^{-1} = K$. And, since $yKy = K$, we have that $gKg^{-1} = y^i hKh^{-1} y^i = y^i K y^i = K$, and hence $K \lhd G$.

Thus, $S, K \lhd G$ and $|S|\,|K| = |G|$ [remember that $K \in \mathcal{S}$, i.e., $|K| = 3$]. Also, $S \cap K = \{1\}$, since $K \neq \{1, x, x^2\}$ [by our choice of K]. Using Proposition 2.8.6, we have that $G \cong S \times K$, i.e.,

$$\boxed{G \cong S_3 \times C_3.}$$

Therefore, there are five groups of order 18 up to isomorphism: C_{18}, $C_3 \times C_6$, D_{18}, the semi-direct product of $C_3 \times C_3$ with C_2 [which we haven't seen before], and $S_3 \times C_3$.

□