The Zero Scalar Curvature Yamabe Problem on Noncompact Manifolds with Boundary

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Dedicated to the memory of Professor José F. Escobar

ABSTRACT. Let (M^n, g), $n \geq 3$ be a noncompact complete Riemannian manifold with compact boundary and f a smooth function on ∂M. In this paper we show that for a large class of such manifolds, there exists a metric within the conformal class of g that is complete, has zero scalar curvature on M, and has mean curvature f on the boundary.

The problem is equivalent to finding a positive solution to an elliptic equation with a non-linear boundary condition with critical Sobolev exponent.

1. INTRODUCTION

The celebrated Riemann Mapping Theorem states that any simply connected region in the plane is conformally diVeomorphic to a disk. This theorem is less successful in higher dimensions since very few domains are conformally diVeomorphic to the ball. Nevertheless, we can still ask whether a manifold with boundary is conformally diVeomorphic to a manifold that resembles the ball, namely to one that has zero scalar curvature and constant mean curvature on its boundary. Escobar studied this problem in [2]. He showed that most compact manifolds with boundary admit such conformally related metrics.

A generalization of this problem is the so-called *prescribed mean curvature problem*. Let (M^n, g), $n \geq 3$ be a manifold with boundary and $f \in C^\infty(\partial M)$.

Problem 1.1. *Does there exist a metric conformally equivalent to g that is complete, scalar flat, and has mean curvature f on ∂M?*

Escobar and Garcia [1] studied this problem on (B^3, δ_{ij}). They proved that a Morse function is the mean curvature of a scalar-flat metric $g \in [\delta_{ij}]$ if it satisfies some Morse inequalities. They paralleled Schoen and Zhang's [8] blow-up

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analysis for the prescribed scalar curvature problem on S^3. In both cases, a general solution is unexpected because of the Kazdan-Warner obstruction [3]. (See [4] for the prescribed Gaussian curvature problem on open 2-manifolds.)

In this paper we address Problem 1.1 on a large class of *noncompact* manifolds with boundary (M^n, g), $n \geq 3$. As a corollary of Theorem 2.5 about PDEs we get the following result.

Theorem 1.2. Any smooth function f on ∂M can be realized as the mean curvature of a complete scalar flat metric conformal to g.

In contrast with the compact case, no topological obstructions on f arise. This is a surprising phenomenon.

2. PRELIMINARIES

Let (M^n, g), $n \geq 3$ be a complete, n-dimensional Riemannian manifold with boundary $\partial M \neq \emptyset$. Denote by $\tilde{g} = u^{4/(n-2)} g$ a metric conformally related to g, where $u > 0$ is a smooth function.

It is a standard fact that the relation between the scalar curvature $R(g)$ of the metric g and the scalar curvature $R(\tilde{g})$ of the metric $\tilde{g}$ is given by

$$R(\tilde{g}) = -\frac{4(n-1)}{n-2} \frac{L_g u}{u^{(n+2)/(n-2)}}, \tag{2.1}$$

where $L_g = \Delta_g - (n-2)R(g)/(4(n-1))$, and Δ_g is the Laplacian calculated with respect to the metric g.

The relation between the mean curvature of the boundary $h(g)$ of the metric g, and the mean curvature of the boundary $h(\tilde{g})$ of the metric $\tilde{g}$ is given by

$$h(\tilde{g}) = \frac{2}{n-2} \frac{B_g u}{u^{n/(n-2)}}, \tag{2.2}$$

where $B_g = \partial/\partial\eta + (n-2)h(g)/2$ and $\partial/\partial\eta$ is the outward-pointing normal derivative on ∂M calculated with respect to the metric g.

Remark 2.1. The exponent $n/(n-2)$ of equation (2.2) is called a *critical exponent* since the Sobolev trace embedding $H^1(M) \hookrightarrow L^q(\partial M)$ ceases to be compact for $q \geq 2(n-1)/(n-2)$. This condition rules out the direct method of minimization to prove existence of solutions.

It follows directly from the above discussion that finding a conformally related metric $\tilde{g} = u^{4/(n-2)} g$ on M that is scalar flat (i.e., has zero scalar curvature) and has prescribed mean curvature f on the boundary is equivalent to finding smooth $u > 0$ on M that satisfies equation (2.1) with $R(\tilde{g}) \equiv 0$ and equation (2.2) with $h(\tilde{g}) \equiv f$.

In this paper we find such u for a more general problem, the so-called *supercritical* equation, in which the critical exponent $n/(n-2)$ of (2.2) is replaced by an arbitrary number $\beta > 1$.

Definition 2.2. Let (M^n, g) be a complete, noncompact Riemannian manifold. On each end E of M, consider the volume of the set obtained by intersecting E with the geodesic ball of radius t centered at some fixed $p \in M$, and denote it by $V_E(t)$. We say that the end E is *large* if

$$\int_1^\infty \frac{t}{V_E(t)}\,dt < \infty.$$

Suppose that the Ricci curvature of M satisfies

$$\mathrm{Ric}_M(x) \geq -(n-1)K(1+r(x))^{-2},$$

where $K \geq 0$ is some constant and $r(x)$ is the distance from x to some fixed point p. By Li and Tam's [6] paper, on any large end E of M there exists a harmonic, non-negative function v_E (a *barrier*), which is asymptotic to 1 on E and it is exactly zero on the boundary of a large ball intersected with the end.

Throughout this paper M will be a manifold that satisfies the above bound on the Ricci tensor.

Definition 2.3. We say that (M, g) is *positive* if it is complete, scalar flat, and has positive mean curvature on the boundary.

Remark 2.4. If (M, g) is positive, it has a positive first eigenvalue for the following problem:

$$\begin{cases} L_g u = 0 & \text{in } M, \\ B_g u = \lambda u & \text{on } \partial M. \end{cases}$$

Conversely, if the first eigenvalue of the above problem is positive, then (M, g) admits a conformally-related scalar flat metric g' that has positive mean curvature on the boundary, but this metric *may not be complete*. Theorem 2.5 still applies for (M, g') provided it has large ends, since completeness is not used in the proof. Nevertheless, the new metric $\tilde{g} = u^{4/(n-2)}g'$, which is scalar flat and has prescribed mean curvature f on the boundary, may also be incomplete.

Theorem 2.5. Let (M, g) be a noncompact positive Riemannian manifold with compact boundary and finitely many ends, all of them large. Let f be a smooth function on ∂M and $\beta > 1$. There exists ε, $\delta > 0$ and a smooth function $\varepsilon \leq u \leq \varepsilon + \delta$ on M with

$$\begin{cases} L_g u = 0 & \text{in } M, \\ B_g u = \dfrac{n-2}{2} f u^\beta & \text{on } \partial M. \end{cases} \tag{2.3}$$

When β is the critical exponent $n/(n-2)$, $\tilde{g} = u^{4/(n-2)}g$ is a complete, scalar flat metric on M, with mean curvature $h(\tilde{g}) \equiv f$.

Remark 2.6. For the $\beta = n/(n-2)$ case, the bound $\varepsilon \le u \le \varepsilon + \delta$ guarantees a complete metric $\tilde{g} = u^{4/(n-2)} g$.

Remark 2.7. Since (M, g) is positive, we have that $L_g \equiv \Delta_g$.

A very important class of examples of positive *noncompact* manifolds with boundary is obtained by removing submanifolds of large codimension out of positive *compact* manifolds with boundary. We refer the reader to the Appendix for more details on the construction.

The proof of Theorem 2.5 is divided in two parts. In Section 3 we prove that an iterative process using sub- and super-solutions converges to a solution of (2.3). In Section 4 we construct the appropriate sub- and super-solutions. Theorem 1.2 follows by choosing $\beta = n/(n-2)$ in Theorem 2.5.

3. METHOD OF SUB- AND SUPER-SOLUTIONS

In this section we adapt a method of sub- and super-solutions to our setting. (See [5] for general properties of sub- and super-solution methods on semilinear elliptic problems.) We begin by proving a form of maximum principle on a compact piece of M that contains ∂M.

Let $u \in C^2(M) \cap C^1(\bar{M})$, and define the operators

$$\begin{aligned} L_\lambda u &:= \Delta_g u - \lambda u && \text{in } M, \\ B_\gamma u &:= \frac{\partial u}{\partial \eta} + \left(\frac{n-2}{2} h_g + \gamma\right) u && \text{on } \partial M, \end{aligned}$$

for λ, $\gamma > 0$ fixed large numbers.

Proposition 3.1 (Maximum Principle). *Let $M_1 \subseteq M$ be a compact piece of M containing ∂M, with smooth boundary $\partial M_1 = \partial M \cup N$. Suppose that $u \in C^2(M_1) \cap C^1(\bar{M}_1)$ satisfies:*

$$\begin{cases} L_\lambda u \ge 0 & \text{in } M_1, \\ B_\gamma u \le 0 & \text{on } \partial M, \\ u \le 0 & \text{on } N. \end{cases}$$

Then $u \le 0$ on M_1.

Proof. Put $w(x) = \max\{0, u(x)\}$, so that $w = 0$ on N. Recall that $\min_{\partial M} h_g > 0$. We get:

$$\begin{aligned} 0 &\le \int_{M_1} (L_\lambda u) w - \int_{\partial M} (B_\gamma u) w \\ &= -\int_{M_1} \nabla u \cdot \nabla w - \lambda \int_{M_1} uw - \gamma \int_{\partial M} uw \\ &= -\int_{M_1} |\nabla w|^2 - \lambda \int_{M_1} w^2 - \gamma \int_{\partial M} w^2. \end{aligned}$$

Hence $w = 0$ in M_1, and so $u \le 0$ in M_1. ◻

Definition 3.2. A sub-solution (resp. super-solution) of equation (2.3) is a function u_- (resp. u^+) in $C^2(M) \cap C^1(\bar{M})$ with

$$\begin{cases} \Delta_g u_- \ge 0 & \text{in } M, \\ B_g u_- - \dfrac{n-2}{2} f u_-^\beta \le 0 & \text{on } \partial M, \end{cases}$$

respectively

$$\begin{cases} \Delta_g u^+ \le 0 & \text{in } M, \\ B_g u^+ - \dfrac{n-2}{2} f (u^+)^\beta \ge 0 & \text{on } \partial M. \end{cases}$$

Theorem 3.3. *If there exist sub- and super-solutions u_-, $u^+ \in C^\infty(M)$ with $0 \le u_- \le u^+ \le c_0$, then there exists a smooth solution u of equation (2.3) with $u_- \le u \le u^+$.*

Proof. We will show that the statement holds for all compact pieces $M_1 \subseteq M$ as above. Then we take pieces converging to M and construct a global solution.

Let M_1 be a compact neighborhood of ∂M in M with smooth boundary $\partial M_1 = \partial M \cup N$. Let λ, $\gamma > 0$ be large enough so that (3.1) admits a solution. Let $u_0 = u^+|_{M_1}$, and define inductively $u_i \in C^2(M_1) \cap C^1(\bar{M}_1)$, $i = 1, 2, \ldots$, to be the unique solution to

$$\begin{cases} L_\lambda u_i = -\lambda u_{i-1} & \text{in } M_1, \\ B_\gamma u_i = \dfrac{n-2}{2} f u_{i-1}^\beta + \gamma u_{i-1} & \text{on } \partial M, \\ u_i = u_{i-1} & \text{on } N. \end{cases} \tag{3.1}$$

Claim. We have $u_- \le \cdots \le u_i \le u_{i-1} \le \cdots \le u^+$.

To prove the claim, we will use induction twice: first, to show that the sequence $\{u_i\}$ is non-increasing and bounded by u^+; then, to prove that it is bounded below by u_-.

To check the first induction step, we see that $L_\lambda(u_1 - u_0) = (\Delta u_1 - \lambda u_1) - (\Delta u_0 - \lambda u_0) = -\lambda u_0 - \Delta u_0 + \lambda u_0 = -\Delta u_0 \ge 0$, because $u_0 = u^+$ is a super-solution.

On the other hand, one has

$$\begin{aligned} B_\gamma(u_1 - u_0) &= \frac{\partial u_1}{\partial \eta} + \left(\frac{n-2}{2}h_g + \gamma\right)u_1 - \frac{\partial u_0}{\partial \eta} - \left(\frac{n-2}{2}h_g + \gamma\right)u_0 \\ &= \frac{n-2}{2}fu_0^\beta + \gamma u_0 - \frac{\partial u_0}{\partial \eta} - \left(\frac{n-2}{2}h_g + \gamma\right)u_0 \\ &= \frac{n-2}{2}fu_0^\beta - \frac{\partial u_0}{\partial \eta} - \frac{n-2}{2}h_g u_0 \\ &\le 0 \end{aligned}$$

since $u_0 = u^+$ is a super-solution. By construction $u_1 - u_0 = 0$ on N.

The maximum principle implies $u_1 \le u_0$ and the first step of the induction follows.

Assume, by induction, that $u_i \le u_{i-1}$.

Then, $L_\lambda(u_{i+1} - u_i) = \Delta u_{i+1} - \lambda u_{i+1} - \Delta u_i + \lambda u_i = -\lambda u_i + \lambda u_{i-1} = \lambda(u_{i-1} - u_i) \ge 0$.

On ∂M we have:

$$\begin{aligned} B_\gamma(u_{i+1} - u_i) &= \frac{n-2}{2}fu_i^\beta + \gamma u_i - \frac{n-2}{2}fu_{i-1}^\beta - \gamma u_{i-1} \\ &= \frac{n-2}{2}f(u_i^\beta - u_{i-1}^\beta) + \gamma(u_i - u_{i-1}). \end{aligned}$$

If f is nonnegative, then the above quantity is nonpositive by induction hypothesis.

On the other hand, if there exists $x \in \partial M$ with $f(x) \le 0$, then by choosing $\gamma > (n-2)\beta\|f\|\,\|u^+\|_{\partial M}/2$ we get

$$\frac{n-2}{2}f(u_i^\beta - u_{i-1}^\beta) + \gamma(u_i - u_{i-1}) \le 0,$$

so the inequality $B_\gamma(u_{i+1} - u_i) \le 0$ follows from the fact that

$$u_{i-1}^\beta - u_i^\beta \le \beta(u^+)^{\beta-1}(u_{i-1} - u_i).$$

Together with the fact that $u_{i+1} - u_i = 0$ on N, it follows by the maximum principle that u_i is non-increasing.

We now show that $u_- \le u_i$.

By hypothesis, $u_- \le u^+ = u_0$. Assume, by induction, that $u_- \le u_{i-1}$. Then $L_\lambda(u_- - u_i) = \Delta u_- - \lambda u_- - \Delta u_i + \lambda u_i = \Delta u_- + \lambda(u_{i-1} - u_-) \ge 0$, by induction hypothesis and the fact that $\Delta u_- \ge 0$.

On ∂M we have

$$\begin{aligned} B_\lambda(u_- - u_i) &= \frac{\partial u_-}{\partial\eta} + \left(\frac{n-2}{2}h_g + \gamma\right)u_- - \left(\frac{n-2}{2}fu_{i-1}^\beta + \gamma u_{i-1}\right) \\ &= B(u_-) + \frac{n-2}{2}f(u_-^\beta - u_{i-1}^\beta) + \gamma(u_- - u_{i-1}) \\ &\le \frac{n-2}{2}f(u_-^\beta - u_{i-1}^\beta) + \gamma(u_- - u_{i-1}). \end{aligned}$$

Should f be positive, this last term would be non-positive by induction hypothesis. Otherwise, $\gamma > \beta(n-2)\|u^+\|_{\partial B}\|f\|/2$ guarantees $B_\lambda(u_- - u_i) \le 0$ since $u_k^\beta - u_-^\beta \le \beta(u^+)^{\beta-1}(u_k - u_-)$ for $k = 1, \ldots, i-1$. The fact that $u_i = u^+$ on N and $u_- \le u^+$ implies $u_- - u_i \le 0$ on N. The claim follows from the maximum principle.

The inequality $u_- \le u_i \le u^+$ in M_1 implies that the sequence u_i is uniformly bounded. From the first equation in (3.1) we conclude that $|\Delta u_i|$ is uniformly bounded as well. Standard elliptic estimates imply that $\|u_i\|_{2,p}$ is uniformly bounded for any $p > 1$, and hence the Sobolev embedding implies that there is a uniform bound for the sequence u_i in the $C^{1,\nu}(\bar{M}_1)$-norm. DiVerentiating the first equation in (3.1) we find that $|\nabla\Delta u_i|$ is uniformly bounded, and L^p elliptic estimates imply that $\|u_i\|_{3,p}$ is uniformly bounded for any $p > n$. The compactness of the embedding $H^{3,p}(M_1) \hookrightarrow C^{2,\nu}(\bar{M}_1)$, $0 < \nu < 1 - n/p$, $p > n$, guarantees the existence of a subsequence of functions u_{i_k} converging to a function $u|_{M_1} \in C^{2,\nu}(\bar{M}_1)$. Because the sequence of functions u_i is monotone we conclude that the whole sequence converges to $u|_{M_1}$. That $u|_{M_1}$ is in $C^\infty(M_1)$ is a standard argument since it solves (2.3) on M_1.

A diagonal procedure on an exhaustion of M by compact pieces like M_1 gives a way to construct a globally defined smooth function $u \in C^\infty(M)$. Clearly $u_- \le u \le u^+$. Also, u is a uniform limit of (a subsequence of) $u|M_1$'s over compact subsets, so it is straightforward to check that it is a solution to equation (2.3). □

4. Existence of Sub- and Super-solutions

We construct an appropriate harmonic function that we will use as a base for our sub- and super-solutions.

Lemma 4.1. There exists $\mu > 0$, and a positive smooth function $\mu \le v \le 1 + \mu$ on M, with

$$\begin{cases} \Delta_g v = 0 & \text{in } M, \\ B_g v < 0 & \text{on } \partial M, \\ v \sim 1 + \mu & \text{near infinity.} \end{cases}$$

Proof. Let $R > 0$ be large. There always exists a positive solution v_R of the homogeneous problem

$$(P_R) \qquad \begin{cases} \Delta v_R = 0 & \text{in } \{x \mid d(x, \partial M) < R\}, \\ v_R = 0 & \text{on } \partial M, \\ v_R = 1 & \text{on } \{x \mid d(x, \partial M) = R\}. \end{cases}$$

A standard argument shows that as $R_i \to \infty$, the sequence v_{R_i} converges uniformly on compact sets to a harmonic function $0 \le v_\infty \le 1$.

Claim. $v_\infty \sim 1$ on each end's infinity.

Let E be an end and $0 \le v_E \le 1$ be a harmonic barrier function that vanishes on the boundary of a large ball intersected with E and is asymptotic to 1 (See [6]). By the maximum principle, v_E is smaller or equal than v_∞. This way, v_∞ is non-zero and asymptotic to 1 on all ends.

We get that $Bv_\infty = \partial/\partial\eta(v_\infty)$, but $\partial/\partial\eta(v_\infty) < 0$ by Hopf's principle, since v_∞ attains its minimum along the boundary (recall that η is the outward-pointing normal of the boundary).

Pick $\mu > 0$ so that $v := v_\infty + \mu$ still satisfies $Bv < 0$. This way, $v \ge \mu > 0$ and v is asymptotic to $1 + \mu$, as desired. □

Proposition 4.2. *For appropriately small constants ε, $\delta > 0$, $u_- := \varepsilon v$ is a sub-solution, and $u^+ := \varepsilon v + \delta$ is a super-solution.*

Proof. Let v be as before. Note that, since v is positive on the boundary, it makes sense to write $\varepsilon v = O(\varepsilon)$ on ∂M. This way, for ε, $\delta > 0$, $\beta > 1$, one has

$$(\varepsilon v + \delta)^\beta = O(\varepsilon^\beta) + O(\delta^\beta) \quad \text{on } \partial M.$$

By definition, $u_- \le u^+$, and both are harmonic. In order for them to be sub- and super-solutions, we just have to check their behavior on the boundary.

Claim. $Bu_- - (n-2)f(u_-)^\beta/2 \le 0$.

Recall that by construction, $Bv < 0$ on the boundary. Hence

$$\begin{aligned} Bu_- - \frac{n-2}{2} f(u_-)^\beta &= \varepsilon Bv - \frac{n-2}{2} f(\varepsilon v)^\beta \\ &\le -\varepsilon \min_{\partial M} |Bv| + \frac{n-2}{2} \max_{\partial M} |f| (\varepsilon v)^\beta \\ &= -O(\varepsilon) + O(\varepsilon^\beta) \le 0 \end{aligned}$$

by taking $\varepsilon > 0$ small enough.

Claim. $Bu^+ - (n-2)f(u^+)^\beta/2 \ge 0$.

We see that

$$\begin{aligned} Bu^+ - \frac{n-2}{2} f(u^+)^\beta &= \varepsilon Bv - \frac{n-2}{2} f(\varepsilon v + \delta)^\beta \\ &\geq -\varepsilon \max_{\partial M} |Bv| + \delta \left(\frac{n-2}{2} h_g \right) - \frac{n-2}{2} \max_{\partial M} |f| (\varepsilon v + \delta)^\beta \\ &= -O(\varepsilon) + O(\delta) - O(\varepsilon^\beta) - O(\delta^\beta). \end{aligned}$$

The line above can be made nonnegative by choosing ε smaller than δ, and δ small (notice the plus sign next to $O(\delta)$).

This way, $0 < \mu\varepsilon \leq u_- \leq u^+ \leq \varepsilon + \mu\varepsilon + \delta$ are sub- and super-solutions respectively. ◻

Proof of Theorem 2.5. The existence of u satisfying (2.3) is granted by applying the above Proposition 4.2 and Theorem 3.3. For the critical case, i.e., $\beta = n/(n-2)$. The completeness of the metric $\bar{g} = u^{4/(n-2)} g$ follows from the lower bound $u \geq u_- \geq \mu\varepsilon > 0$. ◻

Appendix A. Construction of Positive Manifolds

We show how to construct a large class of noncompact complete positive manifolds with boundary. Basically, these examples come from removing "small" submanifolds from positive *compact* manifolds with boundary.

Remark A.1. Positivity of compact manifolds is equivalent to positivity of the first eigenvalue of problem (2.3), since completeness is not an issue. A compact manifold with boundary is positive if and only if its *Yamabe constant* is positive (see [2]).

Let $(N^n, \bar{g})$, $n \geq 3$, be a positive compact manifold with boundary. Consider a collection of submanifolds $\Sigma = \bigcup_{i=1}^k \Sigma_i^{n_i}$, where each Σ_i is a submanifold in the interior of N of dimension $0 \leq n_i \leq (n-2)/2$; put $\Sigma_i = \{p_i\}$ whenever $n_i = 0$.

We will construct a metric $g = u^{4/(n-2)} \bar{g}$ on $M = N \setminus \Sigma$, that is complete, scalar flat, and has positive mean curvature on the boundary. Also, (M, g) will have large ends and will remain positive.

For $p \in \operatorname{int}(N)$ let $G_p > 0$ denote the Green's function for the conformal Laplacian on $(N, \bar{g})$, which always exists and satisfies $L_{\bar{g}} G_p = \delta_p$ and $B_{\bar{g}} G_p = 0$. This way, for $c > 0$, $G_p + c$ satisfies

$$\begin{aligned} L_{\bar{g}}(G_p + c) &= \delta_p, \\ B_{\bar{g}}(G_p + c) &= c\frac{n-2}{2} h_{\bar{g}} > 0 \end{aligned}$$

since $(N, \bar{g})$ is positive.

By a construction on the Appendix of Schoen and Yau's paper [7] which involves the Green's function, one can find, for each Σ_i of positive dimension, positive functions G_i that are singular on Σ_i and satisfy $L_{\bar{g}} G_i = 0$ on $N \setminus \Sigma_i$.

A simple argument like that of Proposition 4.2 shows that for appropriate coeYcients $a_i > 0$, $c > 0$ the function

$$u = \sum_{\{i|n_i>0\}} a_i G_i + \sum_{\{i|n_i=0\}} a_i G_{p_i} + c$$

is singular on Σ and satisfies $L_{\bar{g}} u = 0$ and $B_{\bar{g}} u > 0$. Therefore $g = u^{4/(n-2)} \bar{g}$ remains positive.

The large codimension of the Σ_i guarantees, via the standard argument in [7], that the singularities of u are strong enough to make $g = u^{4/(n-2)} \bar{g}$ complete with large ends.

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