Note

# Combinatorial constructions associated to the dynamics of one-sided cellular automata

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## Abstract

In this paper we study combinatorial constructions which lead to one-sided invertible cellular automata with different dynamical behavior: equicontinuity, existence of equicontinuous points but not equicontinuity, sensitivity and expansivity. In particular, we provide a simple characterization of the class of equicontinuous invertible one-sided cellular automata and we construct families of expansive one-sided cellular automata.
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## 1. Introduction

In recent years the complex behavior of one-dimensional cellular automata has been studied from the topological dynamics point of view. That is, cellular automata are viewed as continuous self-maps of compact metric spaces or standard *topological dynamical systems*. In general, the complex behavior of topological dynamical systems has been formalized using some basic dynamical properties (see [7]), the most classical

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ones are: (1) topological transitivity (existence of a dense orbit), (2) periodic points are dense and (3) sensitive dependence on initial conditions (there is a constant $\varepsilon$ such that for any point $x$ in the system and $\delta > 0$ there is a point within a distance $\delta$ of $x$ and a time $n \in \mathbb{N}$ such that after $n$ iterations of $x$ and such a point their distance is larger than $\varepsilon$). It turns out that conditions (1) and (2) imply (3) [8]. In the case of cellular automata particular properties hold: transitivity is a sufficient condition for sensitivity [5,8], the absence of sensitivity is equivalent to the existence of an equicontinuous point. In each of these two classes (sensitive cellular automata and cellular automata with equicontinuous points) we can distinguish extreme dynamical phenomena. On one hand, equicontinuous cellular automata (all the points are equicontinuous) and on the other hand positively expansive non-invertible cellular automata (the distance between the positive orbits of arbitrary two points of the system becomes larger than $\varepsilon$, where $\varepsilon$ is a fixed constant) and expansive invertible cellular automata (the distance between the orbits of arbitrary two points of the system becomes larger than $\varepsilon$, where $\varepsilon$ is a fixed constant). Equicontinuous cellular automata exhibit a very simple behavior: every orbit is ultimately periodic with the same period and preperiod [11]. The class of positively expansive cellular automata has been studied in [1,2,11,14] obtaining almost a full description of the dynamics. The case of invertible expansive one-sided cellular automata has been considered in [3,15,16,17], where a rich description of their dynamics is given, but there is still open whether they are topologically conjugate to a full shift. General dynamical properties characterizing cellular automata out of these three extreme classes are not known. Nevertheless, some results have been obtained in the classification of elementary cellular automata and permutative cellular automata in [4] (also called permutive cellular automata), and linear cellular automata in [13,10].

The purpose of this paper is to study combinatorial aspects of the dynamics of invertible cellular automata acting on $A^{\mathbb{N}}$ which could provide a way to construct and recognize cellular automata in the classes described above. First, we consider the class of equicontinuous one-sided cellular automata. Using the combinatorial characterization of invertible one-sided cellular automata given in [3] we determine the way they are constructed. We use the same construction to give examples of invertible cellular automata which are sensitive to initial conditions but not expansive, of invertible cellular automata with equicontinuous points but which are not equicontinuous and families of expansive invertible cellular automata.

Now we introduce the main definitions and notations. The reader can look at [12] for more information in symbolic dynamics or [6] for topics in topological dynamics.

Let $A$ be a finite alphabet. Denote by $A^*$ the set of finite sequences or *words* on $A$ (including the empty word) and by $|w|$ the length of a word $w \in A^*$. A *language* $L$ is a subset of $A^*$; let $L_n$ be the set of words of length $n$ of $L$.

Put $K = \mathbb{N}$ or $\mathbb{Z}$. Consider the set $A^K$ of one- or two-sided infinite sequences or *configurations* $x = (x_i)_{i \in K}$, where $x_i \in A$. For $i \leqslant j$ in $K$ put $x(i,j) = x_i \ldots x_j$. Given $u \in A^*$ with $|u| = k \geqslant 1$, define the periodic configuration $u^\infty \in A^{\mathbb{N}}$ by $u^\infty(nk, nk + k - 1) = u$, $n \in \mathbb{N}$. Also, given $x \in A^{\mathbb{N}}$ and $w = w_0 \ldots w_{k-1} \in A^*$ we denote by $wx$ the point $y \in A^{\mathbb{N}}$ defined by $y(0, k-1) = w$ and $y_{k+i} = x_i$, $i \geqslant 0$.

Endow the set of configurations with the product topology and the *shift* $\sigma : A^K \to A^K$, $\sigma(x) = (x_{i+1})_{i \in K}$. The family of cylinder sets

$$[w]_i = \{x \in A^K : x(i, i + |w| - 1) = w\},$$

where $w \in A^*$ and $i \in K$, is a fundamental basis of closed and open (clopen) sets of $A^K$. In this topology $A^K$ is a compact metric space; the distance can be defined by $d(x, y) = \sum_{i \in K} d_i(x, y) 2^{-|i|}$, where $d_i(x, y) = 1$ if $x_i \neq y_i$ and 0 otherwise. The endomorphism $(A^K, \sigma)$ is called *one-sided full shift* when $K = \mathbb{N}$ and *full shift* when $K = \mathbb{Z}$.

A *subshift* is a closed shift invariant subset $S$ of $A^K$. The language associated to $S$ is $L(S) = \{w \in A^* : \exists x \in S, \ \exists i \in K, \ x(i, i + |w| - 1) = w\}$. It is well known that $S$, whether one- or two-sided, is completely described by its language.

A (one-dimensional) *cellular automaton* (or CA) can thus be defined abstractly as a continuous $\sigma$-commuting self-map of a full shift $A^K$, that is, it commutes with the shift map. Cellular automata over $A^{\mathbb{N}}$ are called *one-sided*; those over $A^{\mathbb{Z}}$ are called *two-sided*. They are part of a particular class of continuous $\sigma$-commuting maps between subshifts. This kind of maps are characterized by using local rules or block maps (see [9]). That is, if $F : S \to T$ is a continuous $\sigma$-commuting map between the (one-sided or two-sided) subshifts $S$ and $T$, then there is a "local rule" $f : L(S)_{r(F)+1} \to L(T)_1$ (in the one-sided case) or $f : L(S)_{2r(F)+1} \to L(T)_1$ (in the two-sided case) such that for $x \in S$ and $i \in K$, $F(x)_i = f(x_i \ldots x_{i+r(F)})$ or $F(x)_i = f(x_{i-r(F)} \ldots x_{i+r(F)})$, respectively. The constant $r(F) \in \mathbb{N}$ is called a radius of $F$ (of course it is not unique). Observe that any one-sided cellular automaton can be extended into a two-sided one; the inverse operation is in general not true.

Let $F : S \to S$ and $G : T \to T$ be continuous $\sigma$-commuting self-maps between subshifts $S$ and $T$. We say $G$ is a factor of $F$ if there is an onto continuous map $\pi : S \to T$ such that $\pi \circ F = G \circ \pi$, that is, the following diagram commutes:

$$
\begin{array}{ccc}
& F & \\
S & \to & S \\
\pi \downarrow & & \downarrow \pi \\
T & \to & T \\
& G &
\end{array}
$$

Moreover, if $\pi$ is 1-to-1 we say that it is a conjugacy between $F$ and $G$ (or that they are topologically conjugate). The standard method we will use in this paper to obtain conjugacies of a given block map $F : S \to S$ is the following. Assume $S \subseteq A^K$ with $K = \mathbb{N}$ or $\mathbb{Z}$. Let $R \in \mathbb{N} \setminus \{0\}$ and define $\pi_R : A^K \to (A^R)^K$ by $\pi_R(x) = (x(iR, (i + 1)R - 1))_{i \in K}$. It is straightforward that $\pi_R$ is a conjugacy between $\sigma^R : A^K \to A^K$ and $\sigma : (A^R)^K \to (A^R)^K$. Since $F$ commutes with the shift, it also commutes with powers of the shift and it is transformed through $\pi_R$ into a continuous shift-commuting self-map $F^{(R)}$ acting on the subshift $S^{(R)} = \pi_R(S)$. If we choose $R = r(F)$, it is immediate to see that $r(F^{(R)}) = 1$, and if $F$ is invertible, we can also choose $R$ such that $r(F^{(R)}) = r((F^{(R)})^{-1}) = 1$.

## 2. Construction of one-sided invertible cellular automata

In this section we recall the construction of one-sided invertible cellular automata given in [3]. All cellular automata studied in this section are one-sided.

Let $F:A^{\mathbb{N}} \to A^{\mathbb{N}}$ be an invertible cellular automaton. The inverse map $F^{-1}$ is also a cellular automaton. By using the standard recoding method described in Section 1 we can suppose that $r(F) = r(F^{-1}) = 1$. That is, there exist $f:A^2 \to A$ and $\tilde{f}:A^2 \to A$ such that for $x \in A^{\mathbb{N}}$ and $i \in \mathbb{N}$, $F(x)_i = f(x_i x_{i+1})$ and $F^{-1}(x)_i = \tilde{f}(x_i x_{i+1})$. Both $f$ and $\tilde{f}$ are left permutative, that is, for all $b \in A$ the maps $f(\cdot\ b):A \to A$ and $\tilde{f}(\cdot\ b):A \to A$ are permutations. In what follows, $f$ and $\tilde{f}$ will always denote the local rules of $F$ and $F^{-1}$, respectively.

We say that a letter $a' \in A$ is a successor of $a \in A$ with respect to $F$, and we use the shorthand notation $a \xrightarrow{F} a'$, if there are $x, x' \in A^{\mathbb{N}}$ such that $F(ax) = a'x'$. Analogously we define $a \xrightarrow{F^{-1}} a'$.

We associate to the invertible cellular automaton $F$ the following relations over $A$: given $a, b \in A$, we will say $a \sim'_F b$ if and only if there are $c, d \in A$ such that $f(ac) = f(bd)$ (that have a common successor), and define $\sim_F$ as the transitive closure of $\sim'_F$. The relation $\sim_{F^{-1}}$ is defined analogously. The following lemmas and propositions can be found in [3].

**Lemma 2.1.** *The relations $\sim_F$ and $\sim_{F^{-1}}$ are equivalence relations.*

For $a \in A$ denote by $c_F(a)$ its equivalence class with respect to $\sim_F$ and by $c_{F^{-1}}(a)$ its equivalence class with respect to $\sim_{F^{-1}}$.

**Lemma 2.2.** *Let $a, b \in A$. If $a \sim_F b$ then $\forall c \in A, \ f(ca) = f(cb)$.*

Let us define the functions $\pi_F : A \to A$ and $\pi_{F^{-1}} : A \to A$ by $\pi_F(a) = f(aa)$, $\pi_{F^{-1}}(a) = \tilde{f}(aa)$ for any $a \in A$. Since $F$ is invertible these functions are permutations.

**Lemma 2.3.** *For all $a, b \in A, \ f(ab) \in \{\pi_F(a') : a' \in c_F(a)\}$.*

We have stated that any invertible cellular automaton $F$ with $r(F) = r(F^{-1}) = 1$ determines a partition of $A$, $\mathscr{P}_F = \{c_F(a): a \in A\}$, such that:
(1) $\forall a, b \in A, \forall b', b'' \in c_F(b), \ f(ab') = f(ab'') \in \pi_F(c_F(a))$,
(2) $f$ is left permutative.
These last two properties characterize invertible one-sided cellular automata.

**Proposition 2.4.** *Let $\mathscr{P}$ be a partition of $A$ and $F:A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a cellular automaton with $r(F) = 1$ satisfying the following conditions:*
(1) *$\pi_F$ is a permutation,*
(2) *$f$ is left permutative,*
(3) *$\forall a \in A, f(a \cdot):A \to A$ is constant on atoms of $\mathscr{P}$,*
(4) *$\forall B \in \mathscr{P}, \ \forall a \in B, \ Im(f(a\ \cdot)) \subseteq \pi_F(B)$.*
*Then $F$ is an invertible cellular automaton with $r(F^{-1}) = 1$.*

The last proposition shows a way to construct all one-sided invertible cellular automata up to topological conjugacy (we only construct cellular automata with $r(F) = r(F^{-1}) = 1$). First, we fix a partition $\mathscr{P}$ of $A$ and after that we complete the local rule of $F$ by following properties (1)–(4). This construction can always be performed. The partition $\mathscr{P}_F$ is a refinement of the partition $\mathscr{P}$ used to construct the cellular automaton $F$.

**Example 2.5.** Let $A = \{0,1,2\}$ and consider the partition $\mathscr{P} = \{\{0\},\{1,2\}\}$. The procedure proposed above provides the following invertible cellular automata with $r(F) = r(F^{-1}) = 1$:

(a) $f(00) = f(01) = f(02) = \pi(0)$, $f(10) = f(11) = f(12) = \pi(1)$, $f(20) = f(21) = f(22) = \pi(2)$,

and

(b) $f(00) = f(01) = f(02) = \pi(0)$, $f(11) = f(12) = \pi(1)$, $f(21) = f(22) = \pi(2)$, $f(10) = \pi(2)$, $f(20) = \pi(1)$,

where $\pi : A \to A$ is a permutation. In example (a) $\mathscr{P}_F = \{\{0\},\{1\},\{2\}\}$, which is finer than $\mathscr{P}$, and in example (b) we obtain $\mathscr{P} = \mathscr{P}_F$.

The equivalence classes in $\mathscr{P}_F$ and $\mathscr{P}_{F^{-1}}$ are related in the following way.

**Lemma 2.6.** *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be an invertible cellular automaton with $r(F) = r(F^{-1}) = 1$. For every $a,b \in A$, $a \sim_F b$ if and only if $\pi_F(a) \sim_{F^{-1}} \pi_F(b)$.*

It is direct from last lemma that $c_{F^{-1}}(a) = \pi_F(c_F(\pi_F^{-1}(a)))$ for any $a \in A$. In particular, the cardinality of $\mathscr{P}_F$ and $\mathscr{P}_{F^{-1}}$ is the same. In Example 2.5(a) and (b) the partitions of $A$ determined by $F^{-1}$ are $\mathscr{P}_{F^{-1}} = \{\{\pi(0)\},\{\pi(1)\},\{\pi(2)\}\}$ and $\mathscr{P}_{F^{-1}} = \{\{\pi(0)\}, \{\pi(1),\pi(2)\}\}$, respectively.

**Example 2.7.** We consider additive invertible cellular automata over a finite Abelian group $(A,+)$, with $r(F) = r(F^{-1}) = 1$. That is, a cellular automaton $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ with local rule $f(x_0 x_1) = \pi_0(x_0) + \pi_1(x_1)$ and $\pi_0 : A \to A$, $\pi_1 : A \to A$ endomorphisms of the group. It is not difficult to prove that $\pi_0$ is an automorphism of $(A,+)$ and $\pi_1 \circ \pi_0^{-1} \circ \pi_1 = 0$. We show that these properties are enough to determine an additive invertible cellular automaton with $r(F) = r(F^{-1}) = 1$.

**Claim.** Let $\pi_0$ and $\pi_1$ be endomorphisms of $(A,+)$ such that $\pi_0$ is 1-to-1 and $Ker(\pi_1 \circ \pi_0^{-1} \circ \pi_1) = A$. Then, the cellular automaton $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ with local rule $f(x_0 x_1) = \pi_0(x_0) + \pi_1(x_1)$ is invertible and $r(F) = r(F^{-1}) = 1$.

**Proof.** Let $x,y,z \in A$ and put $\pi_0(x) + \pi_1(y) = a$, $\pi_0(y) + \pi_1(z) = b$. From these equalities and $\pi_1 \circ \pi_0^{-1} \circ \pi_1(z) = 0$ we get $\pi_1(y) = \pi_1 \circ \pi_0^{-1}(b)$ and $x = \pi_0^{-1}(a) - \pi_0^{-1} \circ \pi_1 \circ \pi_0^{-1}(b)$. This is exactly the local rule of $F^{-1} : \tilde{f}(ab) = \pi_0^{-1}(a) - \pi_0^{-1} \circ \pi_1 \circ \pi_0^{-1}(b)$. □

In this example the relation $\sim_F$ can be easily characterized. If $a,b \in A$, then $a \sim_F b$ if and only if $\exists c \in A$, $\pi_0(a) = \pi_0(b) + \pi_1(c)$, and $c_F(a) = a + Im(\pi_0^{-1} \circ \pi_1)$.

## 3. Equicontinuous one-sided cellular automata

Let us recall that a homeomorphism $T$ of a compact metric space $(X, d)$ is equicontinuous if for any $\varepsilon > 0$ there is $\delta > 0$ such that $d(T^n(x), T^n(y)) \leqslant \varepsilon$ whenever $d(x, y) \leqslant \delta$ for any $n \in \mathbb{N}$. It is not difficult to see that a cellular automaton $F: A^{\mathbb{N}} \to A^{\mathbb{N}}$ is equicontinuous if there exists $M \in \mathbb{N}$ such that for any $x, y \in A^{\mathbb{N}}$ if $x(0, M) = y(0, M)$ then $F^n(x)(0, r(F) - 1) = F^n(y)(0, r(F) - 1)$ for any $n \in \mathbb{N}$. By using the standard recoding methods described before it is possible to prove that any equicontinuous invertible cellular automaton is topologically conjugate to a cellular automaton with $r(F) = r(F^{-1}) = 1$ and $M = 1$. For this reason in this section we reduce our study to this case.

The dynamics of equicontinuous invertible cellular automata is well known: every orbit is periodic of constant period [11]. Let us remark that all surjective equicontinuous cellular automata are invertible. In fact, it is not hard to prove that every orbit of a surjective equicontinuous cellular automaton is periodic of constant period, which in turn implies that the restriction of the cellular automaton to the set of shift periodic points is injective. Therefore it is invertible. Using this fact and the characterization of invertible cellular automata given in the previous section we will study the way that equicontinuous cellular automata are constructed.

Let us begin with a basic observation. Proposition 2.4 implies that inside each atom of $\mathscr{P}_F$ the local rule of the invertible cellular automaton is a one-block map which is defined by the permutation $\pi_F$. To define equicontinuous cellular automata we have to control the relations between classes in $\mathscr{P}_F$ that appear under the action of the automaton. The following proposition shows a natural way to produce an equicontinuous map from any invertible cellular automaton. Let $\hat{\pi}_F$ be the extension of $\pi_F$ to $A^{\mathbb{N}}$ as a one-block map.

**Proposition 3.1.** *Let $F: A^{\mathbb{N}} \to A^{\mathbb{N}}$ be an invertible cellular automaton with $r(F) = r(F^{-1}) = 1$. Then $\hat{F} = \hat{\pi}_F^{-1} \circ F$ is equicontinuous.*

**Proof.** Since $\hat{\pi}_F$ is an invertible one-block map, $\hat{F}$ is invertible and $r(\hat{F}) = r(\hat{F}^{-1}) = 1$. Also, it is straightforward that $\pi_{\hat{F}}$ is the identity. Therefore, since for each $a' \in c_{\hat{F}}(a)$ the set $\{\hat{f}(a'b) | b \in A\} \subseteq \pi_{\hat{F}}(c_{\hat{F}}(a)) = c_{\hat{F}}(a)$, successors of elements in a class remain in the same class. Notice that $c_F(a) = c_{\hat{F}}(a)$ for every $a \in A$. On the other hand, we know from Lemma 2.2 that $\hat{f}(ab) = \hat{f}(ab')$ if $b$ and $b'$ are in the same equivalence class. Hence the trajectory of a point $x \in A^{\mathbb{N}}$ with respect to the 0-coordinate partition is completely determined from $x(0, 1)$. In fact, once the equivalence class of $x_1$ is determined then $\hat{F}^i(x)(0) = \hat{f}(\hat{F}^{i-1}(x)(0)b)$ for any $b \in c_{\hat{F}}(x_1)$ and $i \in \mathbb{N} \setminus \{0\}$. This last fact proves that $\hat{F}$ is equicontinuous. $\quad\square$

The cellular automaton $\hat{F}$ is called "the skeleton" of $F$. A simple sufficient condition for equicontinuity based on the skeleton map is provided by the following lemma.

**Lemma 3.2.** *Let $F: A^{\mathbb{N}} \to A^{\mathbb{N}}$ be an invertible cellular automaton with $r(F) = r(F^{-1}) = 1$. If $\hat{F}$ commutes with $\hat{\pi}_F$ then $F$ is equicontinuous.*

**Proof.** From Proposition 3.1 we know that $\hat{F}$ is equicontinuous. Since $\hat{\pi}_F$ is a one-block map, it is also equicontinuous. Finally if two equicontinuous cellular automata commute, their composition is also equicontinuous. $\square$

The last condition is not necessary for equicontinuity. To see it consider the equicontinuous invertible cellular automaton with $r(F) = r(F^{-1}) = 1$ defined on $\{0, 1, 2, 3, 4, 5\}^{\mathbb{N}}$ by: $f(0a) = 3$, $f(0b) = 4$, $f(1a) = 4$, $f(1b) = 5$, $f(2a) = 5$, $f(2b) = 3$, $f(3a) = 0$, $f(3b) = 2$, $f(4a) = 1$, $f(4b) = 1$, $f(5a) = 2$, $f(5b) = 0$, where $a \in \{0, 1, 2\}$ and $b \in \{3, 4, 5\}$. Therefore, $\pi_F(\hat{f}(04)) = 4$ and $\hat{f}(\pi_F(0)\pi_F(4)) = 5$ which shows that $\hat{F}$ does not commute with $\hat{\pi}_F$.

Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be a cellular automaton with $r(F) = r(F^{-1}) = 1$. We define the partition $\tilde{\mathscr{P}}_F$ of $A$ as the set of equivalence classes of the relation in $A$, $a \mathscr{R} b$ if and only if $f(\cdot a) = f(\cdot b)$ (as functions from $A$ to $A$). By Lemma 2.2 we have $\mathscr{P}_F \succcurlyeq \tilde{\mathscr{P}}_F$.

We will define a new partition $\mathscr{P}_F^*$ which lies between $\mathscr{P}_F$ and $\tilde{\mathscr{P}}_F$. If $B, C \in \mathscr{P}_F$ we say $B \sim C$ if there is $a \in A$ and points $x, y \in A^{\mathbb{N}}$ such that $F^i(ax)(0) \in B$ and $F^i(ay)(0) \in C$ for some $i \in \mathbb{N}$. Let us generalize the last relation by transitivity. That is, if $B_0, \ldots, B_n$ are atoms in $\mathscr{P}_F$ such that $B_0 \sim B_1, \ldots, B_{n-1} \sim B_n$, then $B_0 \sim^* B_n$. Finally, the atoms of $\mathscr{P}_F^*$ are unions of elements in $\mathscr{P}_F$ related by $\sim^*$. Consequently $\mathscr{P}_F \succcurlyeq \mathscr{P}_F^*$. The following proposition shows that under equicontinuity $\mathscr{P}_F^* \succcurlyeq \tilde{\mathscr{P}}_F$.

**Proposition 3.3.** *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be an invertible equicontinuous cellular automaton with $r(F) = r(F^{-1}) = 1$ and equicontinuity constant $M = 1$. If $a \in A$, $B \in \mathscr{P}_F^*$ and $b, b' \in B$ then $f(ab) = f(ab')$.*

**Proof.** Since $F$ is reversible, then $f$ is left-permutive, that is, given $a'$ in $A$, for all $b' \in A$, there is a (unique) letter $b \in A$ such that $f(ba') = b'$. Since $F$ is reversible, so is $F^i$ for any $i \in \mathbb{N}$, and the following property holds: given $x \in A^{\mathbb{N}}$ and $b' \in A$, there is $b \in A$ such that $F^i(bx) = b'F^i(x)$.

Let $a \in A$ and $x, y \in A^{\mathbb{N}}$, $x \neq y$. First we prove that for any $b' \in A$ and $i \in \mathbb{N}$, $f(b'F^i(ax)(0)) = f(b'F^i(ay)(0))$. Under the assumption of equicontinuity (with $M = 1$), for any $b \in A$, and any integer $i \in \mathbb{N}$, we have $F^i(bax)(0) = F^i(bay)(0)$. Now, combining the two first points: given any letter $b' \in A$, and any integer $i \in \mathbb{N}$, there is a letter $b \in A$ such that $F^i(bax) = b'F^i(ax)$. If $F$ is equicontinuous (with $M = 1$), then $F^i(bax)$ and $F^i(bay)$ have the same starting letter, so such a letter $b$ also satisfies: $F^i(bay) = b'F^i(ay)$. By equicontinuity, we also have: $F(i + 1)(bax)(0) = F(i + 1)(bay)(0)$, which can be rewritten as $f(b'F^i(ax)(0)) = f(b'F^i(ay)(0))$, which proves the desired property.

Let $B, C \in \mathscr{P}_F$ verifying $B \sim C$. By definition, there exist $a \in A$ and points $x, y \in A^{\mathbb{N}}$ such that for some $i \in \mathbb{N}$, $F^i(ax)(0) \in B$ and $F^i(ay)(0) \in C$. Therefore, by the property proved above, we have $f(a'b) = f(a'b')$ for any $a' \in A$ and $b, b' \in B \cup C$. Since atoms $D \in \mathscr{P}_F^*$ are unions of sets in $\mathscr{P}_F$ which are related by $\sim^*$, then $f(a'b) = f(a'b')$ for any $a' \in A$ and $b, b' \in D$. $\square$

**Proposition 3.4.** *Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be an invertible equicontinuous cellular automaton with $r(F) = r(F^{-1}) = 1$ and equicontinuity constant $M = 1$. For any $B \in \mathscr{P}_F^*$ there is a unique $C \in \mathscr{P}_F^*$ such that, $\forall a \in B, \forall b \in A$, $a \xrightarrow{F} b \Rightarrow b \in C$.*

**Proof.** Let $B \in \mathscr{P}_F^*$. We prove the proposition in several steps. Take $\tilde{B} \in \mathscr{P}_F$ such that $\tilde{B} \subseteq B$. If $a, a' \in \tilde{B}$ then there are $a_0 = a, a_1, \ldots, a_N = a' \in A$ such that $a_i \sim_F a_{i+1}$ for $i \in \{0, \ldots, N-1\}$. In particular, $a_i$ and $a_{i+1}$ have a common successor $b_i \in A$. On the other hand, using the definition of $\mathscr{P}_F^*$, we deduce that successors of a fixed element of $A$ must belong to the same atom in $\mathscr{P}_F^*$. Therefore, we have determined a unique $C \in \mathscr{P}_F^*$ such that $b_0, \ldots, b_{N-1} \in C$. We conclude that

$$\forall a \in \tilde{B}, \ \forall b \in A, \ a \xrightarrow{F} b \ \Rightarrow \ b \in C. \tag{$*_0$}$$

Now, consider $\tilde{B}_1, \tilde{B}_2 \subseteq B$, $\tilde{B}_1, \tilde{B}_2 \in \mathscr{P}_F$, such that $\tilde{B}_1 \sim \tilde{B}_2$. By definition, there exist $a \in A$ and $x, y \in A^{\mathbb{N}}$ such that $F^i(ax)(0) \in \tilde{B}_1$ and $F^i(ay)(0) \in \tilde{B}_2$ for some $i \in \mathbb{N}$. Then the atom of $\mathscr{P}_F$ containing $F^{i+1}(ax)(0)$ and the atom containing $F^{i+1}(ay)(0)$ must be related by $\sim$. This means that $F^{i+1}(ay)(0)$ and $F^{i+1}(ax)(0)$ belong to the same element of $\mathscr{P}_F^*$. Combining this fact and property $(*_0)$ we conclude,

$$\forall a \in \tilde{B}_1 \cup \tilde{B}_2, \ \forall b \in A, \ a \xrightarrow{F} b \Rightarrow b \in C, \tag{$*_1$}$$

where $C \in \mathscr{P}_F^*$ is uniquely determined.

To finish let us take $\tilde{B}_1 \sim^* \tilde{B}_2$, where $\tilde{B}_1, \tilde{B}_2 \subseteq B$, $\tilde{B}_1, \tilde{B}_2 \in \mathscr{P}_F$. Then, $\tilde{B}_1 = C_0 \sim C_1$, $C_1 \sim C_2, \ldots, C_{L-1} \sim C_L = \tilde{B}_2$, for a sequence $(C_i)_{i=0}^L \subseteq \mathscr{P}_F$. If we use property $(*_1)$ recursively we conclude there is a unique $C \in \mathscr{P}_F^*$ such that

$$\forall a \in \tilde{B}_1 \cup \tilde{B}_2, \ \forall b \in A, \ a \xrightarrow{F} b \ \Rightarrow \ b \in C.$$

This last property proves the proposition.  $\square$

Observe that for $B \in \mathscr{P}_F^*$, $a \in B$ and $b \in A$, the condition $a \xrightarrow{F} b$ is equivalent to $b \in \pi_F(B)$. Therefore, from last proposition we deduce that $\pi_F(B) \subseteq C$ for a unique $C \in \mathscr{P}_F^*$. Moreover, since the partition is finite and the map $F$ is invertible, we have $\pi_F(B) = C$.

**Theorem 3.5.** *An invertible cellular automaton* $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ *with* $r(F) = r(F^{-1}) = 1$ *is equicontinuous with equicontinuity constant* $M = 1$ *if and only if there is a partition* $\mathscr{P}$ *of* $A$ *such that* $\mathscr{P}_F \succcurlyeq \mathscr{P} \succcurlyeq \tilde{\mathscr{P}}_F$ *and* $\pi_F(B) \in \mathscr{P}$ *for any* $B \in \mathscr{P}$.

**Proof.** The fact that the condition is necessary is a direct consequence of Proposition 3.3 and the last comment. To prove the condition is sufficient first remark that $f(ab) = f(ab')$ for any $b, b' \in B$ and $a \in A$, where $B \in \mathscr{P}$. In fact, $\mathscr{P} \succcurlyeq \tilde{\mathscr{P}}_F$.

Now, take $x \in A^{\mathbb{N}}$ and $B \in \mathscr{P}$ such that $x(1) \in B$. By hypothesis, there is a sequence $(B_i)_{i \in \mathbb{N}} \subseteq \mathscr{P}$ such that $B = B_0$ and $\pi_F(B_i) = B_{i+1}$ for $i \in \mathbb{N}$. Therefore, $(F^i(y)(0))_{i \in \mathbb{N}} = (F^i(x)(0))_{i \in \mathbb{N}}$ for any $y \in A^{\mathbb{N}}$ such that $y(0, 1) = x(0, 1)$. This fact proves that $F$ is equicontinuous with equicontinuity constant $M = 1$.  $\square$

The last theorem provides a way to discover invertible one-sided equicontinuous cellular automata but it does not give an algorithm to decide such a property since condition is only valid for $M = 1$.

## 4. Examples

In this section we use the procedure given in Section 2 to construct some examples of invertible cellular automata which are not equicontinuous. Namely, we give a cellular automaton having equicontinuous points but which is not equicontinuous, we construct a family of expansive cellular automata and finally we provide a cellular automaton which is sensitive to initial conditions but it is not expansive. For completeness we first prove that expansive cellular automata are sensitive to initial conditions. Recall that a one-sided invertible cellular automaton $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is expansive if there is $\varepsilon > 0$ such that for every pair of points $(x, y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$ there exists $i \in \mathbb{Z}$ for which $d(F^i(x), F^i(y)) > \varepsilon$ ($d$ is a metric in $A^{\mathbb{N}}$). Equivalently, given $(x, y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$ there exists $i \in \mathbb{Z}$ such that $F^i(x)(0, r(F) - 1) \neq F^i(y)(0, r(F) - 1)$. Thus an expansive invertible cellular automaton is topologically conjugate to the subshift $S_F = \{(F^i(x)(0, r(F) - 1))_{i \in \mathbb{Z}} : x \in A^{\mathbb{N}}\}$.

**Proposition 4.1.** *Let* $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ *be an expansive invertible cellular automaton, then* $F$ *is sensitive to initial conditions.*

**Proof.** By recoding the local rule we can assume $r(F) = r(F^{-1}) = 1$. Let us suppose $F$ is not sensitive. Therefore, there is an equicontinuous point $x \in A^{\mathbb{N}}$. That is, for a certain constant $N \in \mathbb{N}$, if $x(0, N) = y(0, N)$, where $y \in A^{\mathbb{N}}$, then $F^i(x)(0) = F^i(y)(0)$ for all $i \in \mathbb{N}$.

Consider the shift periodic points $\bar{x} = x(0, N)^\infty$ and $\bar{y} = (x(0, N)w)^\infty$, where $w \in A^*$ is chosen so that $\bar{x} \neq \bar{y}$. From the invertibility of $F$ it follows that $\bar{x}$ and $\bar{y}$ are also periodic for $F$. Fix $p \in \mathbb{N}$ such that $F^p(\bar{x}) = \bar{x}$ and $F^p(\bar{y}) = \bar{y}$. Since $x$ is equicontinuous, we have that $F^i(\bar{x})(0) = F^i(\bar{y})(0)$ for $i \in \{0, \ldots, p - 1\}$, which implies, by periodicity, that $F^i(\bar{x})(0) = F^i(\bar{y})(0)$ for any $i \in \mathbb{Z}$. Therefore, by expansiveness, $\bar{x}$ must be equal to $\bar{y}$ which is a contradiction.  □

### 4.1. An invertible cellular automaton having equicontinuous points which is not equicontinuous

Let us consider the alphabet $A = \{0, 1, 2\}$. We define a cellular automaton $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by the local rule $f : A^2 \to A$ given by

$$f(0, 0) = f(0, 2) = 0, \quad f(0, 1) = 1,$$
$$f(2, 0) = f(2, 2) = 1, \quad f(2, 1) = 0,$$
$$f(1, 0) = f(1, 1) = f(1, 2) = 2.$$

This map satisfies the conditions of Proposition 2.4, therefore it is an invertible cellular automaton such that $r(F) = r(F^{-1}) = 1$. The partitions defined by $F$ are $\mathscr{P}_F = \{\{0, 2\}, \{1\}\}$ and $\mathscr{P}_{F^{-1}} = \{\{0, 1\}, \{2\}\}$.

In order to determine the dynamics of $F$ we will study the subshift

$$S_F = \{y \in \{0, 1, 2\}^{\mathbb{Z}} : \exists \, x \in A^{\mathbb{N}}, \ (F^i(x)(0))_{i \in \mathbb{Z}} = y\}.$$

**Property 1.** *For every* $n \geqslant 0$, $120^{2n+1}12 \notin L(S_F)$.

**Proof.** We prove the property by induction on $n$. It follows from the definition of $F$ that the word $12012 \notin L(S_F)$, then the property is true for $n = 0$. Assume, by induction hypothesis, that for $n \geqslant 0$ the property holds. Suppose $120^{2n+3}12 \in L(S_F)$. Therefore, we can find $x \in A^{\mathbb{N}}$ such that $(F^i(x)(0))_{i=0}^{2n+6} = 120^{2n+3}12$. On the other hand, we have that $f(2a) = 0 \Leftrightarrow a = 1$ and $f(0a) = 1 \Leftrightarrow a = 1$. Therefore, $F(x)(1) = 1$, $F^2(x)(1) = 2$, $F^{2n+4}(x)(1) = 1$, $F^{2n+5}(x)(1) = 2$ and for $i \in \{3, \dots, 2n + 3\}$, $F^i(x)(1) = 0$. We have constructed a word $120^{2n+1}12 \in L(S_F)$, which contradicts our hypothesis and consequently $120^{2n+3}12 \notin L(S_F)$.  $\square$

**Property 2.** *Let $x \in A^{\mathbb{N}}$ such that $x_i x_{i+1} = 11$ for some $i \in \mathbb{N}$. Then $x$ is an equicontinuous point of $F$.*

**Proof.** It is enough to prove that $x = 11y \in A^{\mathbb{N}}$ is an equicontinuous point of $F$. From Property 1 we know that $F^{2k+1}(x)(1) \in \{0, 2\}$ for any $k \in \mathbb{Z}$, then $F^{2k}(x)(0)F^{2k+1}(x)(0) = 12$. This fact proves the property.  $\square$

We have proved that $F$ has equicontinuous points. To see that $F$ is not equicontinuous just notice that $x = (0)_{i \in \mathbb{N}}$ is not an equicontinuous point for $F$. In fact, $F^i(x)(0) = 0$ and $F^i(0^i 1 y)(0) = 1$ for any $y \in A^{\mathbb{N}}$, $i \in \mathbb{N}$.

## 4.2. Some classes of expansive cellular automata

Let $\mathscr{A} = \{0, \dots, p - 1\}$ with $p \geqslant 2$. We define for $n \geqslant 1$ alphabets $\mathscr{A}_1 = \mathscr{A} \times \mathscr{A}$, $\mathscr{A}_n = \mathscr{A}_{n-1} \times \mathscr{A}_1 = \mathscr{A}_1^n$. The natural projection of $\mathscr{A}_n$ onto $\mathscr{A}_{n-1}$ is denoted by $\phi_n$.

We will define a family of expansive (invertible) cellular automata $F_n : \mathscr{A}_n^{\mathbb{N}} \to \mathscr{A}_n^{\mathbb{N}}$, $n \geqslant 1$, with $r(F_n) = r(F_n^{-1}) = 1$. The construction is made recursively and it starts from examples in [3]. There, the main purpose was to show classes of expansive cellular automata topologically conjugate to full shifts. This last fact remains the main conjecture for expansive one-sided cellular automata. The new examples have the same nice dynamical behavior as those given in [3].

We recall the construction of an expansive cellular automaton $F_1 : (\mathscr{A}_1)^{\mathbb{N}} \to (\mathscr{A}_1)^{\mathbb{N}}$ given in [3]. First we define partitions $\mathscr{P}_{F_1}$ and $\mathscr{P}_{F_1^{-1}}$, then permutation $\pi_{F_1} : \mathscr{A}_1 \to \mathscr{A}_1$ and finally we determine the local rule $f_1 : \mathscr{A}_1^2 \to \mathscr{A}_1$. Let $\mathscr{P}_{F_1}$ and $\mathscr{P}_{F_1^{-1}}$ be partitions of $\mathscr{A}_1$ such that

(1) $|\mathscr{P}_{F_1}| = |\mathscr{P}_{F_1^{-1}}| = p$ and $\forall C \in \mathscr{P}_{F_1}$, $\forall D \in \mathscr{P}_{F_1^{-1}}$, $|C| = |D| = p$;

(2) $\forall C \in \mathscr{P}_{F_1}$, $\forall D \in \mathscr{P}_{F_1^{-1}}$, $|C \cap D| = 1$.

Let $\pi_{F_1} : \mathscr{A}_1 \to \mathscr{A}_1$ be a permutation verifying:

(3) $\forall C \in \mathscr{P}_{F_1}$, $\pi_{F_1}(C) \in \mathscr{P}_{F_1^{-1}}$.

Finally choose a local rule $f_1$ such that:

(4) $f_1$ is left-permutative;

(5) $\forall a \in \mathscr{A}_1$, $f_1(a \cdot) : \mathscr{A}_1 \to \mathscr{A}_1$ is constant over atoms of $\mathscr{P}_{F_1}$;

(6) $\forall a \in \mathscr{A}_1$, $Succ_{F_1}(a) = \{f_1(ab) : b \in \mathscr{A}_1\} = c_{F_1^{-1}}(\pi_{F_1}(a)) = \pi_{F_1}(c_{F_1}(a))$.

Conditions (1)–(6) determine the cellular automaton $F_1$ which is expansive invertible with $r(F_1) = r(F_1^{-1}) = 1$. Moreover, it is topologically conjugate to the full shift $\mathscr{A}^{\mathbb{Z}}$ [3].

For $n \geqslant 2$ we define recursively an invertible cellular automaton $F_n : \mathscr{A}_n^{\mathbb{N}} \to \mathscr{A}_n^{\mathbb{N}}$ with local rule $f_n : \mathscr{A}_n^2 \to \mathscr{A}_n$ and $r(F_n) = r(F_n^{-1}) = 1$. Recall $\pi_{F_n} : \mathscr{A}_n \to \mathscr{A}_n$, $\pi_{F_n}(a) = f_n(aa)$. As before, first we define partitions,

$$\mathscr{P}_{F_n} = \{\{(a,i,j) \in \mathscr{A}_n : i \in \mathscr{A}\} : a \in \mathscr{A}_{n-1}, \ j \in \mathscr{A}\},$$

$$\mathscr{P}_{F_n^{-1}} = \{\{(a,i,j) \in \mathscr{A}_n : j \in \mathscr{A}\} : a \in \mathscr{A}_{n-1}, \ i \in \mathscr{A}\},$$

and we consider a new one

$$\mathscr{B}_{F_n} = \{\{(a,i,j) : a \in B, i,j \in \mathscr{A}\} : B \in \mathscr{B}_{F_{n-1}}\}.$$

If $n = 1$ partition $\mathscr{B}_{F_1}$ coincides with $\mathscr{P}_{F_1}$.

The cellular automaton $F_n$ is defined in such a way that the following conditions hold:

(C1) $f_n$ is left-permutative;

(C2) $\pi_{F_n}$ is a permutation;

(C3) $\forall a \in \mathscr{A}_n, B \in \mathscr{B}_{F_n}, |f_n(aB)| = 1$, that is, $f_n(a \cdot) : \mathscr{A}_n \to \mathscr{A}_n$ is constant on atoms of $\mathscr{B}_{F_n}$;

(C4) $\forall C \in \mathscr{P}_{F_n}, \pi_{F_n}(C) \in \mathscr{P}_{F_n^{-1}}$;

(C5) $\forall (a,i,j) \in \mathscr{A}_n, \ Succ_{F_n}((a,i,j)) = c_{F_n^{-1}}(\pi_{F_n}((a,i,j))) = \pi_{F_n}(c_{F_n}((a,i,j)))$;

(C6) $\forall (a,i,j) \in \mathscr{A}_n, \ \pi_{F_n}((a,i,j)) = (b,k,l) \Rightarrow b \in \pi_{F_{n-1}}(c_{F_{n-1}}(a))$;

(C7) $\forall a \in \mathscr{A}_{n-1}, \forall i \in \mathscr{A}, \forall j, j' \in \mathscr{A}, j \neq j', \ \phi_n(\pi_{F_n}((a,i,j))) \neq \phi_n(\pi_{F_n}((a,i,j')))$.

From these conditions we deduce that for any $a^* \in \mathscr{A}_{n-1}, j^* \in \mathscr{A}$ there exist unique $b^* \in \mathscr{A}_{n-1}$, $i^* \in \mathscr{A}$ such that $\pi_{F_n}(\{(a^*,i,j^*) : i \in \mathscr{A}\}) = \{(b^*,i^*,j) : j \in \mathscr{A}\}$. Moreover, $j^*$ is uniquely determined from $a^*$ and $b^*$. Also, given $B \in \mathscr{B}_{F_n}$ there exist $C \in \mathscr{B}_{F_{n-1}}$ such that $B = C \times \mathscr{A} \times \mathscr{A}$.

The cellular automaton $F_n$ is invertible with $r(F_n) = r(F_n^{-1}) = 1$ since it satisfies conditions of Proposition 2.4. In fact, the local rule of $F_n^{-1}$ can be explicitly determined. Let $(b,k,l), (b',k',l') \in \mathscr{A}_n$, then there exist $a \in \mathscr{A}_{n-1}$, $j \in \mathscr{A}$ such that $Succ_{F_n^{-1}}((b,k,l)) = \{(a,i,j) : i \in \mathscr{A}\}$ and for any $i \in \mathscr{A}$ there is a unique $B(i) \in \mathscr{B}_{F_n}$ with $f_n((a,i,j)B(i)) = \{(b,k,l)\}$. Then

$$\tilde{f}_n((b,k,l)(b',k',l')) = (a,i,j) \quad \text{if and only if} \quad (b',k',l') \in \pi_{F_n}(B(i)).$$

Now we prove that each $F_n$ is expansive and topologically conjugate to $\mathscr{A}^{\mathbb{Z}}$.

**Lemma 4.2.** *Let* $B_0, \ldots, B_m \in \mathscr{B}_{F_n}$, $m \in \mathbb{N}$, $n \geqslant 1$. *Then there is* $x \in \mathscr{A}_n^{\mathbb{N}}$ *such that* $(F_n^i(x))_0 \in B_i$, $i \in \{0, \ldots, m\}$.

**Proof.** We prove this property by induction on $n$. The case $n = 1$, $m \in \mathbb{N}$, follows from [3]. Let $n \geqslant 1$ and $m \in \mathbb{N}$. Clearly if $B_0 \in \mathscr{B}_{F_{n+1}}$ there is $x \in \mathscr{A}_{n+1}^{\mathbb{N}}$ such that $x_0 \in B_0$. Therefore, the lemma is true for $n \geqslant 1$, $m = 0$. Assume the lemma holds for pairs $(n,m')$, $m' \in \mathbb{N}$, and $(n+1, m')$, $m' < m \in \mathbb{N}$. We prove it also holds for $n + 1$ and $m$.

Let $B_0, \ldots, B_m \in \mathscr{B}_{F_{n+1}}$ and $C_0, \ldots, C_m \in \mathscr{B}_{F_n}$ such that $B_i = C_i \times \mathscr{A} \times \mathscr{A}$, $i \in \{0, \ldots m\}$. By induction hypothesis there is $y \in \mathscr{A}_n^{\mathbb{N}}$ such that $y_0 \in C_0$, $(F_n(y))_0 \in C_1, \ldots, (F^m(y))_0 \in C_m$. By construction of $F_{n+1}$, there exist $(i_k, j_k) \in \mathscr{A} \times \mathscr{A}$, $k \in \{0, \ldots, m\}$, such that for all $k \in \{1, \ldots, m\}$, $a_k = ((F_n^k(y))_0, i_k, j_k) \in Succ_{F_{n+1}}((F_n^{k-1}(y))_0, i_{k-1}, j_{k-1}))$ (in fact,

$j_0, \ldots, j_{m-1}, i_1, \ldots, i_m$ are uniquely defined). Also, by construction, there are unique $\bar{B}_0, \ldots, \bar{B}_{m-1} \in \mathscr{B}_{F_{n+1}}$ such that $f_{n+1}(a_k \bar{B}_k) = \{a_{k+1}\}$, $k \in \{0, \ldots, m-1\}$.

Finally, by hypothesis, there is $z \in \mathscr{A}_{n+1}^{\mathbb{N}}$ such that $(F_{n+1}^k(z))_0 \in \bar{B}_k$, $k \in \{0, \ldots, m-1\}$. So $\bar{z} = a_0 z$ verifies $(F_{n+1}^k(\bar{z}))_0 \in B_k$, $k \in \{0, \ldots, m\}$. $\quad\square$

**Lemma 4.3.** *Let $n \geqslant 1$ and $B_{-n}, \ldots, B_{n-1} \in \mathscr{B}_{F_n}$. There exists a unique $a \in B_0$ such that for every $a_{-n} \in B_{-n}, \ldots, a_{n-1} \in B_{n-1}$ with $a_{k+1} \in Succ_{F_n}(a_k)$, $k \in \{-n, \ldots, n-2\}$, it holds $a_0 = a$.*

**Proof.** We prove the statement by induction on $n$. The case $n = 1$ follows from [3]. Assume the property holds for $n \geqslant 1$ and let us prove it is also true for $n + 1$. Let $B_{-n-1}, \ldots, B_n \in \mathscr{B}_{F_{n+1}}$ and $C_{-n-1}, \ldots, C_n \in \mathscr{B}_{F_n}$ such that $B_k = C_k \times \mathscr{A} \times \mathscr{A}$, $k \in \{-n-1, \ldots, n\}$. Take for $k \in \{-n-1, \ldots, n\}$, $a_k = (b_k, i_k, j_k) \in B_k$ with $a_{k+1} \in Succ_{F_{n+1}}(a_k)$. By construction of $F_{n+1}$, $b_{k+1} \in Succ_{F_n}(b_k)$, $k \in \{-n-1, \ldots, n-1\}$. Therefore, by induction hypothesis, $b_{-1}, b_0, b_1$ are uniquely determined from $B_{-n}, \ldots, B_{n-1}$. Now, by property (C7) in the construction, $j_{-1}$ and $j_0$ are uniquely determined, and given $b_{-1}$ and $j_{-1}$, by property (C6) in the construction, $i_0$ is uniquely determined. We conclude $a_0$ is uniquely determined. $\quad\square$

**Corollary 4.4.** *Let $n \geqslant 1$ and $x, y \in \mathscr{A}_n^{\mathbb{N}}$. If $(F_n^k(x))_0 = (F_n^k(y))_0$ for $k \in \{-n, \ldots, n\}$ then $y_1 = x_1$.*

**Proof.** Let $k \in \{-n, \ldots, n-1\}$. By definition of $F_n$ there exist a unique $B_k \in \mathscr{B}_{F_n}$ such that $f_n((F_n^k(x))_0 B_k) = f_n((F_n^k(y))_0 B_k) = \{(F_n^{k+1}(x))_0\} = \{(F_n^{k+1}(y))_0\}$. We remark $(F^k(x))_1, (F^k(y))_1 \in B_k$ and $(F^{k+1}(x))_1 \in Succ_{F_n}((F^k(x))_1)$, $(F^{k+1}(y))_1 \in Succ_{F_n}((F^k(y))_1)$. Then, by Lemma 4.3, $x_1 = y_1$. $\quad\square$

**Proposition 4.5.** *$F_n$ is an invertible, expansive cellular automaton which is topologically conjugate to the full shift $\mathscr{A}^{\mathbb{Z}}$.*

**Proof.** From Corollary 4.4 we deduce that for any $x \in \mathscr{A}_n^{\mathbb{N}}$ the sequence $(F_n^k(x)(0))_{k \in \mathbb{Z}}$ determines uniquely $x$, then $F_n$ is expansive. In addition, the map $\psi_n : \mathscr{A}_n^{\mathbb{N}} \to S_{F_n}$, $\psi_n(x) = ((F_n^k(x))_0)_{k \in \mathbb{Z}}$, is a conjugacy between $(\mathscr{A}_n^{\mathbb{N}}, F_n)$ and $(S_{F_n}, \sigma)$, where

$$S_{F_n} = \{(F_n^k(x)(0))_{k \in \mathbb{Z}} : x \in \mathscr{A}_n^{\mathbb{N}}\}.$$

To conclude, remark that by Lemma 4.2 and Corollary 4.4 the map $\rho_n : \mathscr{A}_n^{\mathbb{N}} \to \mathscr{B}_{F_n}^{\mathbb{Z}}$, $\rho_n(x) = (B_k)_{k \in \mathbb{Z}}$, where $(F^k(x))_0 \in B_k$, is also a conjugacy between $(\mathscr{A}_n^{\mathbb{N}}, F_n)$ and $(\mathscr{B}_{F_n}^{\mathbb{Z}}, \sigma)$. Hence $S_{F_n}$ is topologically conjugate to the fullshift $\mathscr{A}^{\mathbb{Z}}$. $\quad\square$

*4.3. Invertible cellular automata which are sensitive to initial conditions but not expansive*

A direct construction to produce invertible cellular automata which are sensitive to initial conditions but non–expansive is achieved by considering the direct product of an expansive cellular automaton by an equicontinuous one. In fact, from Proposition 4.1 we

know that expansive cellular automata are sensitive to initial conditions, so the direct product of an expansive cellular automaton by an equicontinuous one will preserve sensitivity. On the other hand, such a direct product cannot be expansive since in any open set of the product space there are points whose orbits are arbitrarily close to each other.

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