Ann Glob Anal Geom (2009) 35:115–131
DOI 10.1007/s10455-008-9125-9

ORIGINAL PAPER

Monotonicity of the Yamabe invariant under connect sum over the boundary

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Received: 6 June 2008 / Accepted: 18 August 2008 / Published online: 5 September 2008

Abstract We show that the Yamabe invariant of manifolds with boundary satisfies a monotonicity property with respect to connected sums along the boundary, similar to the one in the closed case. A consequence of our result is that handlebodies have maximal invariant.

Keywords Manifold with boundary · Positive scalar curvature · Yamabe invariant

1 Introduction, main results, and sketch of proof

The Yamabe invariant of a closed manifold is a smooth invariant. It is a minimax of the total scalar curvature functional over the space of unit volume metrics on the manifold. More precisely, one computes the infimum of the functional over a fixed conformal class, and then the supremum of the infimums is computed over the set of all conformal classes.

The Yamabe invariant has been extensively studied. It is known that whenever the minimax is achieved the manifold admits an Einstein metric, which is an important question in geometry. When the invariant is positive it is very difficult to determine whether it is achieved or not. Furthermore, very few invariants have been computed in the positive case.

The analysis of the Yamabe invariant can also be carried out in the realm of manifolds with boundary. The corresponding quantity to study is a quotient of the energy, which in this case is the total scalar curvature plus the total mean curvature of the boundary, divided by a normalization term. Since one can choose to normalize with respect to the total volume of the manifold or the total area of its boundary, two possible definitions for the invariant appear.

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Let (M^n, g) be a compact manifold with boundary. Proceeding in analogy with the closed case, we define the (conformally invariant) **Yamabe constant** of (M, g) by

$$Y_\lambda(M, [g]) = \inf_{\tilde{g}\in[g]} \frac{E(\tilde{g})}{N_\lambda(\tilde{g})}, \tag{1}$$

where $E(\tilde{g})$ is the energy of the metric $\tilde{g}$ and $\lambda \in \{0, 1\}$ determines whether the normalization N_λ is with respect to the volume of $\tilde{g}$ ($\lambda = 1$) or the area of the boundary of $\tilde{g}$ ($\lambda = 0$). It is a well known fact that $-\infty \leq Y_\lambda(M^n, [g]) \leq Y_\lambda(S^n_+, [g_0])$, where g_0 denotes the round metric on the hemisphere.

Existence of minimizers of the above quotient have been studied by Escobar in [5,6], where he proved that solutions exist in most cases (See also [4,9,10]). A standard argument shows that, when writing Eq. 1 in terms of conformal factors of conformally related metrics (see Eqs. 5, 6), solutions of the Euler-Lagrange equations correspond to conformally related metrics that solve the Yamabe problem on (M, g). This is, conformal metrics with constant scalar curvature and minimal boundary (in the case $\lambda = 1$), and scalar flat metrics with boundary having constant mean curvature (whenever $\lambda = 0$).

The **Yamabe invariant** of a smooth manifold with boundary is defined as

$$\sigma_\lambda(M) = \sup_{[g]\in\mathcal{C}} Y_\lambda(M, [g]), \tag{2}$$

where $\mathcal{C}$ is the set of all smooth conformal classes of metrics on M. From the considerations above it follows that $-\infty < \sigma_\lambda(M^n) \leq \sigma_\lambda(S^n_+)$.

In this paper we prove the following result.

Theorem 1.1 *Let M_1, M_2 be smooth n-manifolds with boundary, $n \geq 3$. Let $M_1\#M_2$ denote their connected sum along the boundary. Then, for $\lambda \in \{0, 1\}$,*

$$\sigma_\lambda(M_1\#M_2) \geq \sigma_\lambda(M_1 \sqcup M_2), \tag{3}$$

where $M_1 \sqcup M_2$ denotes the disjoint union of M_1, M_2 and

$$\sigma_\lambda(M_1 \sqcup M_2) = \begin{cases} -(|\sigma_\lambda(M_1)|^{\frac{n}{2}} + |\sigma_\lambda(M_2)|^{\frac{n}{2}})^{\frac{2}{n}} & \sigma_\lambda(M_1), \sigma_\lambda(M_2) \leq 0, \\ \min\{\sigma_\lambda(M_1), \sigma_\lambda(M_2)\} & \text{otherwise.} \end{cases} \tag{4}$$

Our result is an extension, to manifolds with boundary, of Kobayashi's classical theorem about monotonicity of the Yamabe invariant over connect sums of closed manifolds [8]. The proof of our result parallels his proof with the added boundary ingredients, plus some approximation results that are adaptations of some found by Akutagawa and Botvinnik [1].

An interesting consequence of our result is the following corollary.

Corollary 1.2 *Handlebodies have maximal invariant. This is, if H^n is the result of attaching a finite number of handles to the n-ball, then*

$$\sigma_\lambda(H) = \sigma_\lambda(S^n_+), \ \lambda \in \{0, 1\}.$$

Sketch of the Proof of Theorem 1.1 The proof of the theorem is divided into two cases: (a) when the Yamabe invariant of both M_1 and M_2 is positive, and (b) when one of the invariants is nonpositive. The idea behind the proof of (a) is to show that if both M_1, M_2 have positive metrics, then after gluing a long enough handle over the boundary of $M_1 \sqcup M_2$ one obtains a manifold whose Yamabe constant is larger than the Yamabe constant of $M_1 \sqcup M_2$ minus a small error. Therefore, the invariant of $M_1\#M_2$ is no less than the invariant of $M_1 \sqcup M_2$. To prove (b) we endow M_1 and M_2 with some particular negative metrics, which can be done by a classic result of Kazdan and Warner. Using some approximation lemmas (found in the appendix) we prove that the Yamabe constant of the connect sum is bounded below by the appropriate quantities in terms of the Yamabe constants of M_1 and M_2.

2 Preliminaries

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$. The energy of $u \in C^\infty(M)$ with respect to g is defined as

$$E_g(u) = \int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dV_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 dA_g. \tag{5}$$

Using the transformation laws for the scalar curvature and the mean curvature, it is standard to note that the energy of u is the total scalar curvature plus the total mean curvature of the conformally related metric $\tilde{g} = u^{4/(n-2)} g$, i.e., $E_g(u) = E(\tilde{g}) = \int_M R_{\tilde{g}} dV + \frac{n-2}{2} \int_{\partial M} h_{\tilde{g}} dA$.

For $\lambda \in \{0, 1\}$, the normalization factor is just

$$N_{g,\lambda}(u) = \lambda \int_M |u|^{\frac{2n}{n-2}} dV_g + (1-\lambda) \left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} dA_g \right)^{\frac{n}{n-1}}, \tag{6}$$

It follows that $N_{g,\lambda}(u) = N_\lambda(\tilde{g}) = \lambda Vol(M, \tilde{g}) + (1-\lambda)(Area(\partial M, \tilde{g})^{n/(n-1)})$, as desired. This way, we get that

$$Y_\lambda(M, [g]) = \begin{cases} \inf \left\{ \frac{E_g(u)}{N_{g,0}(u)} : u \not\equiv 0 \text{ on } \partial M \right\} & \text{if } \lambda = 0, \\ \inf \left\{ \frac{E_g(u)}{N_{g,1}(u)} : u \not\equiv 0 \text{ on } M \right\} & \text{if } \lambda = 1. \end{cases}$$

Outline of the Proof of Theorem 1.1 The simple argument below shows how Eq. 4 of the theorem is satisfied. The proof of Eq. 3 is split into two cases. In Sect. 3 we deal with the case when both $\sigma_\lambda(M_1)$ and $\sigma_\lambda(M_2)$ are positive, where we prove that Eq. 3 follows from the (more general) Handle Monotonicity Theorem (Theorem 3.1). In Sect. 4 we study the case when at least one of $\sigma_\lambda(M_1)$, $\sigma_\lambda(M_2)$ is nonpositive.

In Sects. 3 and 4 we use a series of approximation arguments. These are modifications of the approximation lemmas that appear in Kobayashi's original paper [8] and Akutagawa and Botvinnik's paper [1]. We include their proofs, for sake of completeness, in the Appendix.

Lemma 2.1 *Let $\lambda \in \{0, 1\}$ and (M_1, g_1), (M_2, g_2) be Riemannian manifolds with boundary. The following holds:*

$$Y_\lambda(M_1 \sqcup M_2, g_1 \sqcup g_2) = \begin{cases} -(|Y_\lambda(M_1, g_1)|^{\frac{n}{2}} + |Y_\lambda(M_2, g_2)|^{\frac{n}{2}})^{\frac{2}{n}} & Y_\lambda(M_1, g_1),\ Y_\lambda(M_2, g_2) \leq 0, \\ \min\{Y_\lambda(M_1, g_1), \quad Y_\lambda(M_2, g_2)\} & \text{otherwise.} \end{cases}$$

Proof Write $E_1(u)$, $V_1(u)$ and $E_2(u)$, $V_2(u)$ for the expressions in Eqs. 5 and 6 integrated over M_1, ∂M_1 and M_2, ∂M_2, respectively. We have that

$$\begin{aligned}
&Y_\lambda(M_1 \sqcup M_2, g_1 \sqcup g_2)\\
&\quad = \inf\left\{E_1(u) + E_2(u) : u \in C^1(\bar{M}) \text{ with } V_1(u) + V_2(u) = 1\right\}\\
&\quad = \inf\left\{V_1(u)^{\frac{n-2}{n}} \frac{E_1(u)}{V_1(u)^{\frac{n-2}{n}}} + V_2(u)^{\frac{n-2}{n}} \frac{E_1(u)}{V_1(u)^{\frac{n-2}{n}}} : V_1(u) + V_2(u) = 1\right\}\\
&\quad = \inf\left\{\alpha^{\frac{n-2}{n}} Y_\lambda(M_1, g_1) + (1-\alpha)^{\frac{n-2}{n}} Y_\lambda(M_2, g_2) : 0 \le \alpha \le 1\right\}\\
&\quad = \begin{cases} -(|Y_\lambda(M_1, g_1)|^{\frac{n}{2}} + |Y_\lambda(M_2, g_2)|^{\frac{n}{2}})^{\frac{2}{n}} & Y_\lambda(M_1, g_1),\ Y_\lambda(M_2, g_2) \le 0,\\ \min\{Y_\lambda(M_1, g_1), Y_\lambda(M_2, g_2)\} & \text{otherwise.}\end{cases}
\end{aligned}$$

□

As a direct consequence of the lemma is the proof of Eq. 4.

Corollary 2.2 *Let M_1^n, M_2^n, $n \ge 3$ be Riemannian manifolds with boundary. The following holds:*

$$\sigma_\lambda(M_1 \sqcup M_2) = \begin{cases} -(|\sigma_\lambda(M_1)|^{\frac{n}{2}} + |\sigma_\lambda(M_2)|^{\frac{n}{2}})^{\frac{2}{n}} & \sigma_\lambda(M_1), \quad \sigma_\lambda(M_2) \le 0,\\ \min\{\sigma_\lambda(M_1), \sigma_\lambda(M_2)\} & \text{otherwise.}\end{cases}$$

3 Proof of the positive case

The proof of Theorem 1.1 is split in two cases: the positive case, which we present here, and the nonpositive case (where at least one of $\sigma_\lambda(M_1)$, $\sigma_\lambda(M_2)$ is nonpositive), which we prove in Sect. 4 below.

Throughout this section we assume that M_1, M_2 are n-dimensional, compact manifolds with boundary ($n \ge 3$) with positive Yamabe invariant. We actually prove here a more general theorem, valid only for the positive case. We show that the Yamabe invariant is monotonic when attaching a handle over the boundary. In particular, from this it follows that all handle-bodies have maximal Yamabe invariant.

Theorem 3.1 (Handle monotonicity) *Let M be a compact manifold with boundary and positive Yamabe invariant ($\sigma_\lambda(M) > 0$). Denote by $\bar{M}$ the manifold obtained by attaching a handle to the boundary of M. Then*

$$\sigma_\lambda(\bar{M}) \ge \sigma_\lambda(M).$$

The handle monotonicity theorem has two immediate consequences.

Proof of Theorem 1.1 in the Positive Case Assuming that $\sigma_\lambda(M_1), \sigma_\lambda(M_2) > 0$, apply Theorem 3.1 to $M_1 \sqcup M_2$ when attaching a handle between a boundary component of M_1 and a boundary component of M_2. Together with Corollary 2.2 this gives

$$\sigma_\lambda(M_1 \# M_2) \ge \sigma_\lambda(M_1 \sqcup M_2) = \min\{\sigma_\lambda(M_1), \sigma_\lambda(M_2)\}.$$

□

Proof of Corollary 1.2 Use the Handle monotonicity theorem on $\bar{B}^n$, taking into consideration that $\sigma_\lambda(\bar{B}^n) = \sigma_\lambda(S_+^n)$ is maximal. □

We now give the proof of the main theorem of this section.

Proof of Theorem 3.1 Fix $\epsilon > 0$ small. Let g' be a metric on M so that $0 < Y_\lambda(M) - \epsilon < Y_\lambda(M, g')$.

Pick $p_1, p_2 \in \partial M$. By Lemma A.8 there is a metric g close to g' so that in a neighborhood of p_1 and p_2, g is isometric to a fixed neighborhood $B_\lambda(q, \epsilon')$ of $q \in \partial S^n_\lambda$, for some $\epsilon' > 0$. The Lemma also gives that $R_g \geq 0$ on M.

There exists a function $\lambda \in C^\infty(M - \{p_1, p_2\})$ which is 1 outside $B_\lambda(q, \epsilon'/2)$ and such that the metric $\tilde{g} = e^\lambda g$ is isometric to the half infinite cylinder $[0, \infty) \times S^{n-1}_+$ around the removed points. For convenience we write

$$(M - \{p_1, p_2\}, \tilde{g}) = [0, \infty) \times S^{n-1}_+ \cup (\tilde{M}, \tilde{g}) \cup [0, \infty) \times S^{n-1}_+,$$

where $\tilde{M}$ is the complement of the two cylinders.

We glue $(\tilde{M}, \tilde{g})$ with $[0, l] \times S^{n-1}_+$ to get a smooth Riemannian manifold $(\bar{M}, g_l)$. This is, $\bar{M}$ is the result of attaching a handle on $\tilde{M}$ connecting neighborhoods of p_1 and p_2. We have the follwing decomposition:

$$(\bar{M}, g_l) = (\tilde{M}, \tilde{g}) \cup [0, l] \times S^{n-1}_+. \tag{7}$$

Since

$$Y_\lambda(\bar{M}, g_l) = \inf_u \frac{\int_{\bar{M}} (|\nabla u|^2 + \frac{n-2}{4(n-1)} R_{g_l} u^2) dV_{g_l} + \frac{n-2}{2} \int_{\partial \bar{M}} h_{g_l} u^2 dA_{g_l}}{\left(\lambda \int_{\bar{M}} |u|^{\frac{2n}{n-2}} dV_{g_l} + (1-\lambda) \left(\int_{\partial \bar{M}} |u|^{\frac{2(n-1)}{n-2}} dA_{g_l} \right)^{\frac{n}{n-1}} \right)^{\frac{n-2}{n}}}$$

we can take a positive function $u_l \in C^\infty(\bar{M})$ such that

$$\int_{\bar{M}} (|\nabla u_l|^2 + R_{g_l} u_l^2) dv_{g_l} + \frac{n-2}{2} \int_{\partial \bar{M}} h_{g_l} u_l^2 d\sigma_l < Y_\lambda(\bar{M}, g_l) + \frac{1}{1+l} \tag{8}$$

and

$$\lambda \int_{\bar{M}} |u_l|^{\frac{2n}{n-2}} dV_{g_l} + (1-\lambda) \left(\int_{\partial \bar{M}} |u_l|^{\frac{2(n-1)}{n-2}} dA_{g_l} \right)^{\frac{n}{n-1}} = 1. \tag{9}$$

Lemma 3.2 *There exists a section, say $\{t_l\} \times S^{n-1}_+$ in the cylindrical part of $\bar{M}$ (see Eq. 7) such that*

$$\int_{\{t_l\} \times S^{n-1}_+} (|\nabla u_l|^2 + u_l^2) dV_l + \int_{\{t_l\} \times S^{n-2}} u_l^2 dA_l < \frac{A}{l},$$

where A is a constant independent of l.

Proof Let

$$A' = -\min\{0, \min_{x \in \partial \bar{M}} h_g(x)\} Area(\partial \tilde{M}, \tilde{g})^{\frac{1}{n-1}} - \min\{0, \min_{x \in \bar{M}} R_g(x)\} Vol(\tilde{M}, \tilde{g})^{\frac{2}{n}}.$$

Using Hölder's inequality on (8) and the fact that $R_{g_l} \geq 0$ outside the cylindrical part of $\bar{M}$ we get

$$\int_{[0,l]\times S_+^{n-1}} \left(|\nabla u_l|^2 + \frac{n-1}{4(n-2)}(n-1)(n-2)u_l^2 \right) dV$$
$$+(n-2)(n-3) \int_{[0,l]\times S^{n-2}} u_l^2 dA$$
$$< Y_\lambda(\bar{M}, g_l) + \frac{1}{1+l} + A'.$$

This way, there is a $t_l \in [0, l]$ such that

$$\int_{\{t_l\}\times S_+^{n-1}} \left(|\nabla u_l|^2 + \frac{n-1}{4(n-2)}(n-1)(n-2)u_l^2 \right) dV + (n-2)(n-3) \int_{\{t_l\}\times S^{n-2}} u_l^2 dA$$
$$< \left(Y_\lambda(\bar{M}, g_l) + \frac{1}{1+l} + A' \right) / l.$$

This proves Lemma 3.2. □

We continue with the proof of Theorem 3.1 as follows: cut off $\bar{M}$ on the section $\{t_l\} \times S_+^{n-1}$ and attach two half-infinite cylinders to it, so $(M - \{p_1, p_2\}, \tilde{g})$ reappears. This time we write it as

$$(M - \{p_1, p_2\}, \tilde{g}) = [0, \infty) \times S_+^{n-1} \cup (\bar{M} - \{t_l\} \times S_+^{n-1}, g_l) \cup [0, \infty) \times S_+^{n-1}.$$

We think of u_l as defined on $\bar{M} - \{t_l\} \times S_+^{n-1}$ and extend it to the whole space $M - \{p_1, p_2\}$ as follows:

Let U_l be a Lipschitz function on $M - \{p_1, p_2\}$ such that

$$U_l = u_l \text{ on } \bar{M} - \{t_l\} \times S_+^{n-1}$$

and

$$U_l(t, x) = \begin{cases} (1-t)\tilde{u}_l(x) & \text{for } (t, x) \in [0, 1] \times S_+^{n-1} \\ 0 & \text{for } (t, x) \in [1, \infty) \times S_+^{n-1} \end{cases}$$

where $\tilde{u}_l = u_l|_{\{t_l\}\times S_+^{n-1}} \in C^\infty(S_+^{n-1})$. Now it is easy to see from (8) and Lemma 3.2 that

$$\int_{M-\{p_1,p_2\}} (|\nabla U_l|^2 + \frac{n-1}{4(n-2)} R_{\tilde{g}} U_l^2) dV_{\tilde{g}} + \frac{n-2}{2} \int_{\partial M} h_{\tilde{g}} U_l^2 dA_{\tilde{g}} < Y_\lambda(\bar{M}, g_l) + \frac{B}{l},$$

where B is a constant independent of l. From Eq. 9 it follows that

$$\lambda \int_{\bar{M}} |U_l|^{\frac{2n}{n-2}} dV_{g_l} + (1-\lambda) \left(\int_{\partial \bar{M}} |U_l|^{\frac{2(n-1)}{n-2}} dA_{g_l} \right)^{\frac{n}{n-1}} > 1.$$

Therefore, we have

$$\inf_U \left\{ \frac{\int_{M-\{p_1,p_2\}} (|\nabla U|^2 + \frac{n-1}{4(n-2)} R_{\tilde{g}} U^2) dV_{\tilde{g}} + \frac{n-2}{2} \int_{\partial M} h_{\tilde{g}} U^2 dA_{\tilde{g}}}{\left(\lambda \int_{M-\{p_1,p_2\}} |U|^{\frac{2n}{n-2}} dV_{\tilde{g}} + (1-\lambda) (\int_{\partial M} |U|^{\frac{2(n-1)}{n-2}} dA_{\tilde{g}})^{\frac{n}{n-1}} \right)^{\frac{(n-2)}{n}}} \right\} \tag{10}$$
$$\leq \sigma_\lambda(\bar{M}),$$

where the infimum is taken over all non-negative Lipschitz functions U with compact support. It follows from the choice of the metric $\tilde{g}$ that the left side of Eq. 10 is equal to $Y_\lambda(M, g)$. Since ϵ was arbitrary we conclude $\sigma_\lambda(M) \leq \sigma_\lambda(\bar{M})$, which completes the proof of Theorem 3.1. □

4 Proof of the nonpositive case

In this section we give a proof of Theorem 1.1 whenever one of $\sigma_\lambda(M_1), \sigma_\lambda(M_2)$ is nonpositive. We first prove some lemmas.

Lemma 4.1 *Let (M^n, g'), $n \geq 3$ be a compact manifold with boundary. Suppose that $-\infty < Y_\lambda(M, g') \leq 0$. Then the following holds:*

($\lambda = 1$) *The scalar curvature R_g of a metric $g \in [g']$ with $h_g \geq 0$ satisfies*

$$\frac{n-2}{4(n-1)}(\min R_g) Vol(M, g)^{2/n} \leq Y_1(M, g),$$

and equality above implies that R_g is a constant.

($\lambda = 0$) *The mean curvature h_g of a metric $g \in [g']$ with $R_g \geq 0$ satisfies*

$$\frac{n-2}{2}(\min h_g) Area(\partial M, g)^{1/(n-1)} \leq Y_0(M, g'),$$

where equality above implies that h_g is a constant.

Proof We first prove the case $\lambda = 1$. Let $g \in [g']$ with $h_g \geq 0$. A standard argument (c.f. [3]) shows that, since $h_g \geq 0$ we must have $\min R_g \leq 0$, or else $Y(M, [g']) > 0$. This way,

$$\begin{aligned} Y_1(M, [g']) &= \inf_{u \not\equiv 0 \text{ in } M} \frac{\int_M (|\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2) dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma}{(\int_M u^{2n/(n-2)} d\sigma)^{(n-2)/n}} \\ &\geq \frac{n-2}{4(n-1)} \inf_{u \not\equiv 0 \text{ in } M} \frac{(\min R_g) \int_M u^2 dv_g}{(\int_M u^{2n/(n-2)} d\sigma)^{(n-2)/n}} \\ &\geq \frac{n-2}{4(n-1)} (\min R_g) Vol(M, g)^{2/n}. \end{aligned}$$

The last line holds by Hölder's inequality. If this inequality were an equality, take a sequence $\{u_n\}$ so that $(E_g(u_n)/N_{g,1}(u_n)) \to Y_1(M, [g'])$. Then $\{u_n\}$ also converges to the infimum of $\frac{\int_M \frac{n-2}{4(n-1)} R_g u^2 dv_g}{(\int_M u^{2n/(n-2)} d\sigma)^{(n-2)/n}}$, and so $||\nabla u_n||_2 \to 0$. We deduce that $\min R_g = (\int_M R_g)/Vol(M, g)$, which in turn implies that R_g is constant.

Proof of the case $\lambda = 0$. Let $g \in [g']$ with $R_g \geq 0$. A similar argument as the one above gives that $\min h_g \leq 0$. This way,

$$\begin{aligned} Y_0(M, [g]) &= \inf_{u \not\equiv 0 \text{ on } \partial M} \frac{\int_M (|\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2) dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma}{(\int_{\partial M} u^{2(n-1)/(n-2)} d\sigma)^{(n-2)/(n-1)}} \\ &\geq \frac{n-2}{2} \inf_{u \not\equiv 0 \text{ on } \partial M} \frac{(\min h_g) \int_{\partial M} u^2 d\sigma}{(\int_{\partial M} u^{2(n-1)/(n-2)} d\sigma)^{(n-2)/(n-1)}} \\ &\geq \frac{n-2}{2} (\min h_g) Area(\partial M, g)^{1/(n-1)}. \end{aligned}$$

The last line is obtained by Hölder's inequality. The case of equality is similar to the cases of equality when $\lambda = 1$. □

The following is a well-known result of Escobar [5,6] which parallels that of Aubin [2] for the closed case.

Theorem 4.2 *Let* $(M^n, g), n \geq 3$ *be a compact manifold with boundary. If* $-\infty < Y_\lambda(M, [g]) < Y_\lambda(B^n, [g_0])$*, then the infimum is achieved at a Yamabe metric. This is,*

($\lambda = 1$) *There exists a function* $u \in C^\infty(M)$ *with* $E_g(u)/N_{g,1}(u) = Y_1(M, [g])$*, and the metric* $\tilde{g} = u^{4/(n-2)}g$ *has zero mean curvature* $h_{\tilde{g}} \equiv 0$ *and constant scalar curvature* $R_{\tilde{g}} = Y_1(M, [g])Vol(M, \tilde{g})^{-2/n}$.

($\lambda = 0$) *There exists a function* $u \in C^\infty(M)$ *with* $E_g(u)/N_{g,0}(u) = Y_0(M, [g])$*, so that the metric* $\tilde{g} = u^{4/(n-2)}g$ *has zero scalar curvature* $R_{\tilde{g}} \equiv 0$ *and constant mean curvature* $h_{\tilde{g}} = Y_0(M, [g])Area(\partial M, \tilde{g})^{-1/(n-1)}$.

A direct consequence of the above theorem together with Lemma 4.1 is the following result.

Corollary 4.3 *Consider* (M, g') *with* $-\infty < Y_\lambda(M, [g']) \leq 0$*. Then*

($\lambda = 1$)

$$\max_{\{g \in [g'] \,:\, h_g \geq 0\}} (\min R_g) Vol(M, g)^{2/n} = Y_1(M, [g']).$$

In particular, if $\sigma_1(M) \leq 0$*, we get that*

$$\sup_{\{g \,:\, g \in [g'],\ h_g \geq 0,\ [g'] \in \mathcal{C}\}} (\min R_g) Vol(M, g)^{2/n} = Y_1(M).$$

($\lambda = 0$)

$$\max_{\{g \in [g'] \,:\, R_g \geq 0\}} (\min h_g) Area(\partial M, g)^{1/(n-1)} = Y_0(M, [g']).$$

In particular, if $\sigma_0(M) \leq 0$*, we get that*

$$\sup_{\{g \,:\, g \in [g'],\ R_g \geq 0,\ [g'] \in \mathcal{C}\}} (\min h_g) Area(\partial M, g)^{1/(n-1)} = Y_0(M).$$

Proof of Theorem 1.1 in the Nonpositive Case We first prove the case $\lambda = 1$. For a real number $a \in \mathbb{R}$, let a_- denote the negative part of a. This is $a_- = \max\{-a, 0\}$.

By Kazdan and Warner's Theorem [7] there are metrics g_1, g_2 on M_1, M_2, respectively, such that

$$\begin{cases} \min R_{g_1} = \min R_{g_2} = -((Y_1(M_1)_-)^{n/2} + (Y_1(M_2)_-)^{n/2} + 2\epsilon)^{2/n}, \\ R_{g_1}(p_1) = R_{g_2}(p_2) = n(n-1) \text{ at some points } p_i \in \partial M_i, \\ h_{g_i} \geq 0 \text{ on } \partial M_i, \\ Vol(M_i, g_i) = \frac{(Y_1(M_i)_-)^{n/2} + \epsilon}{(Y_1(M_1)_-)^{n/2} + (Y_1(M_2)_-)^{n/2} + 2\epsilon} \text{ for } i = 1, 2 \end{cases}$$

where $\epsilon > 0$ is an arbitrary number. By Lemma A.10 there exists a metric g of $M_1 \# M_2$ such that

$$h_g \geq 0,\ \ \min R_g = -((Y_1(M_1)_-)^{n/2} + (Y(M_2)_-)^{n/2} + 2\epsilon)^{2/n},$$

and

$$Vol(M_1\#M_2, g) < 1 + \epsilon.$$

Hence, from Corollary 4.3 we get

$$Y_1(M_1\#M_2) \geq -((Y_1(M_1)_-)^{n/2} + (Y_1(M_2)_-)^{n/2})^{2/n},$$

which ends the proof of the first case.

Proof of the case $\lambda = 0$. Let g_i be metrics on $M_i,\ i = 1, 2$ such that

$$\begin{cases} \min h_{g_1} = \min h_{g_2} = -((Y(M_1)_-)^{n-1} + (Y(M_2)_-)^{n-1} + 2\epsilon)^{1/(n-1)}, \\ R_{g_i} \geq 0 \text{ in } M_i, \\ h_{g_i}(p_i) \geq 0,\ R_{g_i}(p_i) = n(n-1), \text{ at some points } p_i \in \partial M_i, i = 1, 2 \\ Area(\partial M_i, g_i) = \frac{(Y(M_i)_-)^{n-1}+\epsilon}{(Y(M_1)_-)^{n-1}+(Y(M_2)_-)^{n-1}+2\epsilon} \text{ for } i = 1, 2 \end{cases}$$

where ϵ is an arbitrary positive number. By Lemma A.10 there exists a metric g on $M_1\#M_2$ such that

$$R_g \geq 0,\ \ \min h_g = -((Y_0(M_1)_-)^{n-1} + (Y_0(M_2)_-)^{n-1} + 2\epsilon)^{1/(n-1)}$$

and

$$Area(M_1\#M_2, g) < 1 + \epsilon.$$

Hence, from Corollary 4.3 we get

$$Y_0(M_1\#M_2) \geq -((Y_0(M_1)_-)^{n-1} + (Y_0(M_2)_-)^{n-1})^{1/(n-1)}.$$

This ends the proof of the nonpositive case of Theorem 1.1. □

Acknowledgements This work was partially supported by NSF grant # DMS-0223098.

Appendix A: Approximation lemmas

This section consists of several approximation lemmas. The results are, essentially, adaptations of those found in Kobayashi [8] and Akutagawa and Botvinnik [1] to fit our scenario. We include the more relevant proofs here.

Lemma 4.4 *For any $\delta > 0$ there exists a smooth non-negative function $0 \leq w_\delta \leq 1$ and a positive constant $\epsilon(\delta),\ 0 < \epsilon(\delta) < \delta$, such that*

(i) $w_\delta(t)|_{[0,\epsilon(\delta)]} \equiv 1,\ w_\delta(t)|_{[\delta,\infty)} \equiv 0,$
(ii) $|t\dot{w}_\delta(t)| < \delta$ *for* $t \geq 0,$
(iii) $|t^2\ddot{w}_\delta(t)| < \delta$ *for* $t \geq 0.$

Proof Elementary. □

Lemma 4.5 *Let $\bar{g}, \tilde{g}$ be two Riemannian metrics on M and $h = \tilde{g} - \bar{g}$. Then $R_{\tilde{g}} - R_{\bar{g}} = P_{\bar{g}}(h) + Q_{\bar{g}}(h)$, where*

$$P_{\bar{g}}(h) = -\Delta_{\bar{g}}(Tr_{\bar{g}}h) + \bar{\nabla}^i\bar{\nabla}^j h_{ij} - \langle h, Ric_{\bar{g}}\rangle_{\bar{g}},$$
$$|Q_{\bar{g}}(h)| \leq c(|\bar{\nabla}h|^2q^3 + |h||\bar{\nabla}h|^2q^2 + (|h||\bar{\nabla}h|^2 + |Ric_{\bar{g}}||h|^2)q).$$

Here, $c = c(n) > 0$ is a constant that depends only on the dimension n of M, and q is a non-negative smooth function that satisfies $q \cdot \tilde{g} \geq \bar{g}$.

Proof A straightforward calculation of writing out the scalar curvature in terms of the metric and its derivatives that can be found in [8]. □

Lemma 4.6 (Fermi coordinates) *Let $(M^n, g), n \geq 3$ be a compact Riemannian manifold with boundary and $o \in \partial M$. There exists Fermi coordinates around o. This is, coordinates (r, x) around o so that $\gamma_x(r) = (r, x)$ is a geodesic and the set $\{(0, x) : x\}$ corresponds to ∂M near o. We have*

(i) $g_{00} = g(\partial_r, \partial_r) = 1,\ g'_{00} = \partial_r g_{00} = 0,$
(ii) $g_{0i} = 0,\ g'_{0i} = \partial_r g_{0i} = 0$ *for* $i > 0,$
(iii) $g_{ij} = (g|_{\partial M})_{ij},\ g'_{ij} = -2A_{ij},\ i, j > 0.$

Proposition A.4 *Let $o \in \partial M$ and $\bar{g}, \tilde{g}$ be metrics on M such that $R_{\bar{g}}(o) = R_{\tilde{g}}(o)$ and $j^1_o\bar{g} = j^1_o\tilde{g}$ (i.e., $\bar{g}$ and $\tilde{g}$ coincide up to first order derivatives on o). Consider Fermi coordinates around o as in Lemma 4.6. Then the family of metrics*

$$\tilde{g}_\delta = \bar{g} + w_\delta(r)(\tilde{g} - \bar{g})$$

has the following properties:

(i) $\tilde{g}_\delta = \bar{g}$ *outside* $B_\delta(o)$,
(ii) $\tilde{g}_\delta = \tilde{g}$ *on* $B_{\epsilon(\delta)}(o)$,
(iii) $h_{\tilde{g}_\delta} = h_{\tilde{g}}$ *on* $B_{\epsilon(\delta)}(o)$,
(iv) $\tilde{g}_\delta \to g$ *in* $C^1(M)$ *as* $\delta \to 0$,
(v) $R_{\tilde{g}_\delta} \to R_{\bar{g}}$ *in* $C^0(M)$ *as* $\delta \to 0$,
(vi) $h_{\tilde{g}_\delta} \to h_{\bar{g}}$ *in* $C^0(\partial M)$ *as* $\delta \to 0$.

Proof The proof of this proposition is similar to Kobayashi's Theorem (Lemma 3.2 of [8]). We proceed as follows: (i) and (ii) are direct and (iii) follows from (ii). To prove (iv) consider the function w_δ from Lemma 4.4. Note that w_δ is supported inside $[0, \delta]$. This way

$$\tilde{g}_\delta - \bar{g} = w_\delta(r)(\tilde{g} - \bar{g}) = O(r^2),$$

and so $\tilde{g}_\delta \to \bar{g}$ in $C^0(M)$. To get the $C^1(M)$ convergence we see that

$$\partial(\tilde{g}_\delta - \bar{g}) = \dot{w}_\delta(r)(\tilde{g} - \bar{g}) + w_\delta(r)\partial(\tilde{g} - \bar{g}).$$

Since $\tilde{g} - \bar{g} = O(r^2)$ and $\partial(\tilde{g} - \bar{g}) = O(r)$, Lemma 4.4 implies $|\partial\tilde{g}_\delta - \partial\bar{g}| \leq |\dot{w}_\delta(r)r|\frac{O(r^2)}{r} + w_\delta(r)O(r) \leq \delta O(\delta) + O(\delta)$, which shows $\partial\tilde{g}_\delta \to \partial\bar{g}$ in $C^0(M)$, as desired.

For (v) we use Lemma 4.5 twice to get

$$\begin{aligned} R_{\tilde{g}_\delta} - R_{\bar{g}} &= P_{\bar{g}}(w_\delta(r)(\tilde{g} - \bar{g})) + Q_{\bar{g}}(w_\delta(r)(\tilde{g} - \bar{g})), \\ 0 &= -R_{\tilde{g}} + R_{\bar{g}} + P_{\bar{g}}(\tilde{g} - \bar{g}) + Q_{\bar{g}}(\tilde{g} - \bar{g}). \end{aligned}$$

Multiplying the second line above by $-w_\delta(r)$ and adding it to the first one we get

$$\begin{aligned} |R_{\tilde{g}_\delta} - R_{\bar{g}}| \leq &\ |P_{\bar{g}}(w_\delta(r)(\tilde{g} - \bar{g})) - w_\delta(r)P_{\bar{g}}(\tilde{g} - \bar{g})| + |Q_{\bar{g}}(w_\delta(r)(\tilde{g} - \bar{g}))| \\ &+ |w_\delta(r)Q_{\bar{g}}(\tilde{g} - \bar{g})| + |w_\delta(r)(R_{\bar{g}} - R_{\tilde{g}})|. \end{aligned}$$

For convenience put

$$\begin{aligned} &T_1 = |P_{\bar{g}}(w_\delta(r)(\tilde{g} - \bar{g})) - w_\delta(r)P_{\bar{g}}(\tilde{g} - \bar{g})|, \quad T_2 = |Q_{\bar{g}}(w_\delta(r)(\tilde{g} - \bar{g}))|, \\ &T_3 = |w_\delta(r)Q_{\bar{g}}(\tilde{g} - \bar{g})|, \quad T_4 = |w_\delta(r)(R_{\bar{g}} - R_{\tilde{g}})|. \end{aligned}$$

Using Lemmas 4.4 and 4.5 and the fact that $j^1_o\tilde{g} = j^1_o\bar{g}$ we get:

$$\begin{aligned}
T_1 &= |P_{\bar{g}}(w_\delta(r)(\tilde{g}-\bar{g})) - w_\delta(r)P_{\bar{g}}(\tilde{g}-\bar{g})|\\
&= |-\Delta_{\bar{g}}(Tr_{\bar{g}}(w_\delta(r)(\tilde{g}-\bar{g}))) + w_\delta(r)\Delta_{\bar{g}}(Tr_{\bar{g}}(\tilde{g}-\bar{g}))\\
&\quad + \bar{\nabla}^i\bar{\nabla}^j(w_\delta(r)(\tilde{g}_{ij}-\bar{g}_{ij})) - w_\delta(r)\bar{\nabla}^i\bar{\nabla}^j(\tilde{g}_{ij}-\bar{g}_{ij})\\
&\quad - \langle w_\delta(r)(\tilde{g}-\bar{g}), Ric_{\bar{g}}\rangle_{\bar{g}} + w_\delta(r)\langle\tilde{g}-\bar{g}, Ric_{\bar{g}}\rangle_{\bar{g}}|\\
&\le c_1(|\ddot{w}_\delta(r)||\tilde{g}-\bar{g}| + |\dot{w}_\delta(r)||\partial(\tilde{g}-\bar{g})|)\\
&= |\ddot{w}_\delta(r)r^2|\frac{O(r^2)}{r^2} + |\dot{w}_\delta(r)r|\frac{O(r)}{r}\\
&= O(\delta),
\end{aligned}$$

where $c_1 > 0$ above is a constant independent of δ. For the second term we get

$$\begin{aligned}
T_2 &= |Q_{\bar{g}}(w_\delta(r)(\tilde{g}-\bar{g}))|\\
&\le c(n)(|\bar{\nabla}(w_\delta(r)(\tilde{g}-\bar{g}))|^2q^3 + |(w_\delta(r)(\tilde{g}-\bar{g}))||\bar{\nabla}(w_\delta(r)(\tilde{g}-\bar{g}))|^2q^2\\
&\quad + (|(w_\delta(r)(\tilde{g}-\bar{g}))||\bar{\nabla}(w_\delta(r)(\tilde{g}-\bar{g}))|^2 + |Ric_{\bar{g}}||(w_\delta(r)(\tilde{g}-\bar{g}))|^2)q).
\end{aligned}$$

Notice that there are only first order derivatives in the above estimate. It is easy to see that $T_2 = O(\delta^2)$. For T_3 we get

$$\begin{aligned}
T_3 &= |w_\delta(r)Q_{\bar{g}}(\tilde{g}-\bar{g})|\\
&\le c(n)|w_\delta(r)|(|\bar{\nabla}(\tilde{g}-\bar{g})|^2q^3 + |\tilde{g}-\bar{g}||\bar{\nabla}(\tilde{g}-\bar{g})|^2q^2\\
&\quad + (|\tilde{g}-\bar{g}||\bar{\nabla}(\tilde{g}-\bar{g})|^2 + |Ric_{\bar{g}}||\tilde{g}-\bar{g}|^2)q)\\
&= O(\delta^2).
\end{aligned}$$

To estimate T_4 we see that

$$T_4 = |w_\delta(r)(R_{\bar{g}} - R_{\tilde{g}})| = O(\delta)$$

since $R_{\bar{g}}(o) = R_{\tilde{g}}(o)$. This way $T_1 + T_2 + T_3 + T_4 = O(\delta)$ and (iv) is proved.

For (vi) we use Lemma 4.6 (iii). We have that $\partial_r(\tilde{g}_\delta)_{ij} = -2\tilde{A}^\delta_{ij}$, so

$$h_{\tilde{g}_\delta} = -(1/2)Tr_{\tilde{g}_\delta}(\tilde{A}^\delta) = -(1/2)Tr_{\tilde{g}_\delta}(\partial_r(\tilde{g}_\delta)_{ij}).$$

Since $\tilde{g}_\delta \to \bar{g}$ in $C^1(M)$, $\partial_r(\tilde{g}_\delta) \to \partial_r\bar{g}$ in $C^0(M)$. Hence, $h_{\tilde{g}_\delta} \to h_{\bar{g}}$ in $C^0(\partial M)$, as desired. This ends the proof. □

Theorem A.5 *Let M be a compact manifold with boundary, $\epsilon_0 > 0$, $o \in \partial M$ and $\bar{g}$ a metric on M. Put $g \equiv \bar{g}|_{\partial M}$ and let $A_{\bar{g}}$ be the second fundamental form of $\bar{g}$ on ∂M. Then there exists a family of metrics $\tilde{g}_\delta$ on M such that*

(i) $\tilde{g}_\delta = \bar{g}$ *in* $M\backslash B_\delta(o)$,
(ii) $h_{\tilde{g}_\delta} = h_{\bar{g}}$ *on* $B_{\epsilon_0}(o, \bar{g})$,
(iii) $\tilde{g}_\delta$ *is conformally equivalent to the metric* $(g - 2rA_{\bar{g}}) + dr^2$ *on* $B_{\epsilon(\delta)}(o, \bar{g})$.
(iv) $\tilde{g}_\delta \to \bar{g}$ *in* $C^1(M)$ *as* $\delta \to 0$,
(v) $R_{\tilde{g}_\delta} \to R_{\bar{g}}$ *in* $C^0(M)$ $\delta \to 0$,
(vi) $h_{\tilde{g}_\delta} \to h_{\bar{g}}$ *in* $C^0(\partial M)$ *as* $\delta \to 0$,

Proof Consider Fermi coordinates (x, r) near o. By Lemma 4.6 we get the following expansion of the metric $\bar{g}$ near o:

$$\bar{g}(x, r) = (g_{ij}(x) - 2rA_{ij}(x) + O(r^2))dx^idx^j + \sum_i O(r^2)drdx^i + dr^2,$$

where $\bar{g}|_{\partial M} = g$ and A_{ij} denotes the second fundamental form of ∂M with respect to $\bar{g}$. Consider the metric

$$\hat{g}(x, r) = (g_{ij}(x) - 2r A_{ij}(x))dx^i dx^j + dr^2$$

which has $j^1_{\partial M}\hat{g} = j^1_{\partial M}\bar{g}$. Now put

$$\tilde{g}(x, r) = u(x, r)^{4/(n-2)}\hat{g}, \tag{11}$$

where

$$u(x, t) = 1 + \frac{1}{2}r^2\phi^2(x), \;\; \phi(x) = -\frac{n-2}{4(n-1)}(R_{\bar{g}} - R_{\hat{g}}). \tag{12}$$

Then $u|_{\partial M} = u(x, 0) = 0,\; \partial_r u(x, 0) = 0$ and so on $\partial M,\; \partial_i u = 0,\; \partial_i\partial_j u = 0$ and $\partial_r u = 0$. This way

$$j^1_o\tilde{g} = j^1_o\bar{g}.$$

The well-known transformation law for the scalar curvature under conformal deformations of the metric gives that $\Delta_{\hat{g}} u = -\frac{n-2}{4(n-1)}(R_{\tilde{g}} u^{(n+2)/(n-2)} - R_{\hat{g}} u)$. In our case $u(x, 0) \equiv 1$ so

$$\Delta_{\hat{g}} u = -\frac{n-2}{4(n-1)}(R_{\tilde{g}} - R_{\hat{g}}) \;\text{ on } \partial M. \tag{13}$$

On the other hand, also on ∂M we have

$$\begin{aligned}
\Delta_{\hat{g}} u &= \nabla^\alpha\partial_\alpha u \\
&= \hat{g}^{\alpha\beta}(\partial_\alpha\partial_\beta u - \hat{\Gamma}^\gamma_{\alpha\beta}\partial_\gamma u) \\
&= \partial_r^2 u + g^{ij}\partial_i\partial_j u - \hat{\Gamma}^0_{00}\partial_r u - \hat{\Gamma}^i_{00}\partial_i u - g^{ij}(\hat{\Gamma}^0_{ij}\partial_r u + \hat{\Gamma}^k_{ij}\partial_k u) \\
&= \partial_r^2 u.
\end{aligned}$$

From this last line and Eq. 13 we get that, on ∂M,

$$-\frac{n-2}{4(n-1)}(R_{\tilde{g}} - R_{\hat{g}}) = \Delta_{\hat{g}} u = \partial_r^2 u = \phi(x) = -\tfrac{n-2}{4(n-1)}(R_{\bar{g}} - R_{\hat{g}}).$$

This way $R_{\tilde{g}} = R_{\bar{g}}$ on ∂M. Now apply Proposition A.4 to $\bar{g},\; \tilde{g}$ to get a family $\tilde{g}_\delta$ that satisfies (i)–(vi). □

Theorem A.6 *Let M be a compact manifold with boundary, $o \in \partial M$ and $\epsilon_0 > 0$. Let $\bar{g}$ be a metric on M with $h_{\bar{g}} = 0$ on $B_{\epsilon_0}(o)$. Write $g \equiv \bar{g}|_{\partial M}$, and put $A_{\bar{g}}$ to be the second fundamental form of ∂M. Then there exists a family of metrics $\tilde{g}_\delta$ such that for δ small enough (compared to ϵ_0) the following holds:*

(i) $\tilde{g}_\delta \equiv \bar{g}$ *on* $M\backslash B_\delta(o)$,
(ii) $\tilde{g}_\delta$ *is conformally equivalent to the metric* $g + dr^2$ *on* $B_{\epsilon(\delta)}(o, \bar{g})$,
(iii) $\tilde{g}_\delta \to \bar{g}$ *in* $C^0(M)$ *as* $\delta \to 0$,
(iv) $R_{\tilde{g}_\delta} \to R_{\bar{g}}$ *in* $C^0(M)$ $\delta \to 0$,
(v) $h_{\tilde{g}_\delta} \to h_{\bar{g}}$ *in* $C^0(\partial M)$ *as* $\delta \to 0$.

Proof The proof of this theorem is similar to Akutagawa and Botvinnik's Approximation Trick (Theorem 4.6 of [1]).

From Eqs. 11 and 12 from the previous proof we may assume that

$$\bar{g} = \left(1 + \frac{1}{2}r^2\phi(x)\right)^{\frac{4}{n-2}} \left((g(x) - 2rA_{\bar{g}}(x)) + dr^2\right)$$

in Fermi coordinates near the point o. Let

$$G_\delta(x, r) = (g(x) - 2r(1 - w_\delta(r))A_{\bar{g}}(x)) + dr^2,$$

and put

$$\tilde{g}_\delta(x, r) = \left(1 + \frac{1}{2}r^2\phi_\delta(x, r)\right)^{\frac{4}{n-2}} G_\delta(x, r),$$

$$\phi_\delta(x, r) = \phi(x) - \frac{3(n-2)}{4(n-1)}(2 - w_\delta(r))w_\delta(r)|A_{\bar{g}}|_g^2.$$

We see that (i) and (ii) come from the construction. For (iii) notice that $\phi_\delta \to \phi$ in $C^0(M)$ and $G_\delta \to \bar{g}$ in $C^0(M)$, and so $\tilde{g}_\delta \to \bar{g}$.

For (iv) a computation shows that near o we have

$$R_{G_\delta} = R_g + 3(1 - w_\delta(r))^2|A_{\bar{g}}|_g^2 - (1 - w_\delta(r))^2 h_{\bar{g}}^2$$
$$-(4\dot{w}_\delta(r) + 2r\ddot{w}_\delta(r))h_{\bar{g}} + O(\delta).$$

Since $h_{\bar{g}} = 0$ on $B_{\epsilon_0}(o)$ we get

$$R_{G_\delta} = R_g + 3(1 - w_\delta(r))^2|A_{\bar{g}}|_g^2 + O(\delta)$$

near o. This way

$$\begin{aligned} R_{\tilde{g}_\delta} &= \left(1 + \frac{1}{2}r^2\phi_\delta(x, r)\right)^{-\frac{n+2}{n-2}} \left(-\frac{4(n-2)}{n-1}\Delta_{G_\delta}\left(1 + \frac{1}{2}r^2\phi_\delta(x, r)\right)\right. \\ &\quad \left. + R_{G_\delta}(1 + \frac{1}{2}r^2\phi_\delta(x, r))\right) \\ &= (1 + O(\delta^2))\left(-\frac{4(n-2)}{n-1}\phi + 3(2 - w_\delta(r))w_\delta(r)|A_{\bar{g}}|_g^2 + R_{G_\delta} + O(\delta)\right) \\ &= R_g + 3(1 - w_\delta(r))^2|A_{\bar{g}}|_g^2 + R_{\bar{g}} - R_g - 3|A_{\bar{g}}|_g^2 + 3(2 - w\delta(r))w_\delta(r)|A_{\bar{g}}|_g^2 \\ &\quad + O(\delta) \\ &= R_{\bar{g}} + O(\delta), \end{aligned}$$

so (iv) follows.

To prove (v) note that near o

$$\left(1 + \frac{1}{2}r^2\phi_\delta\right)(x, 0) = 1, \quad \partial_r\left(1 + \frac{1}{2}r^2\phi_\delta\right)(x, 0) = 0.$$

The usual transformation law for the mean curvature under conformal deformations gives that $\partial_r u = \frac{n-2}{2}(h_{\tilde{g}_\delta}u^{n/(n-2)} - h_{G_\delta}u)$. This way, $h_{\tilde{g}_\delta} = h_{G_\delta}$ near o. Since $h_{G_\delta} = h_{\bar{g}} = 0$ near o the result follows. □

Lemma A.7 *Let $(M, \bar{g})$ be a manifold with boundary and $o \in \partial M$ such that, in Fermi coordinates (r, x), $\bar{g} = g(x) + dr^2$ near o, and so that $R_{\bar{g}}(o) = n(n-1)$. Let $e \in S^n_+$ be an* *equatorial point in the boundary of* (S^n_+, g_n)*, so that in polar normal coordinates around e, the round metric* $g_n = dt^2 + \sin^2(t)g_{n-1}$*. Here,* g_{n-1} *denotes the round metric on* S^{n-1}*.*

Define $\tilde{g} = dt^2 + \sin^2(t)g_{n-1}(y)$ *around* $o \in M$*, where* (t, y) *are polar normal coordinates centered at o. Then:*

(*i*) $R_{\bar{g}}(o) = R_{\tilde{g}}(o)$,
(*ii*) $j^1_o(\bar{g}) = j^1_o(\tilde{g})$.

Proof Part (i) is direct since $R_{\tilde{g}}(o) = R_{g_n}(e) = n(n-1)$. For (ii) we note that all first order partial derivatives of $\bar{g}$ vanish at o since the boundary ∂M has zero second fundamental form near o (see Lemma 4.6 about Fermi coordinates). The same thing holds at e, since the equator in S^n is totally geodesic. □

Lemma A.8 *Let* $(M, \bar{g})$ *be a manifold with* $Y_\lambda(M, \bar{g}) > 0$*,* $p_1, p_2 \in \partial M$ *and* $\delta > 0$*. Then there exist a metric* $\tilde{g}$ *on M and* $0 < \epsilon < \delta$ *such that*

(*i*) $|Y(M, [\bar{g}]) - Y(M, [\tilde{g}])| < \delta$,
(*ii*) $\tilde{g}|_{B_\epsilon(p_i)}$ *around* p_i *is isometric to a neighborhood of* $e \in \partial S^n_+$ *inside* S^n_+*, for* $i = 1, 2$.
(*iii*) $R_{\tilde{g}} \geq 0$ *on M,*
(*iv*) $h_{\tilde{g}} = 0$ *outside a small neighborhood of* p_i*.*

Proof We start by finding $\tilde{g}$. Since the Yamabe constant of $(M, \bar{g})$ is positive we may assume that $\bar{g}$ has positive scalar curvature and minimal boundary. Using Theorem A.6 we can approximate $\bar{g}$ by a metric $\hat{g}$ that is conformal to a product near each p_i. Lemma A.7 together with Proposition A.4 imply that there exists a metric $\check{g}$ on M that is close to $\hat{g}$ and satisfies (ii). Apply Proposition A.4 to $\bar{g}$ and $\check{g}$ to produce a metric $\tilde{g}$ that coincides with $\check{g}$ near p_i and with equals $\bar{g}$ elsewhere. Now $\tilde{g}$ is close to the original $\bar{g}$ so (i) follows. (ii) holds by construction. Since $R_{\tilde{g}}$ is close to $R_{\bar{g}} > 0$, it is non-negative, so (iii) holds. (iv) holds since all constructions took place in a small neighborhood of p_i and $\bar{g}$ had minimal boundary. □

Let (S^n_+, g_n), $n \geq 3$ be the Euclidean unit hemisphere and r the intrinsic distance relative to g_n from a point e on the boundary, so that $g_n = dr^2 + \sin^2 r g_{n-1}$, where g_{n-1} is the standard metric on the unit $n-1$ hemisphere. For an interval $I \subseteq [0, \pi]$ denoted by $A(I)$ the region $A(I) = \{x \in S^n_+ : r(x) \in I\}$.

Lemma A.9 *For any* $\epsilon_1 > 0,\ 0 < \epsilon_2 < \pi$ *there exists a positive function* $f = f(r)$ *on* S^n_+ *such that*:

(*i*) $|R_{g'} - n(n-1)| < \epsilon_1$*, where* $g' = f^{-2}g_n$*,*
(*ii*) $h_{g'} = 0$,
(*iii*) $|Vol(S^n_+, g') - 2Vol(S^n_+, g_n)| < \epsilon_1$,
(*iv*) $|Area(\partial S^n_+, g') - 2Area(\partial S^n_+, g_{n-1})| < \epsilon_1$,
(*v*) $f(r) = 1$ *for* $r > \epsilon_2$ *and* $(A([0, \epsilon_2')), g')$ *is isometric to* $(A((\epsilon_2, \pi]), g') = (A((\epsilon_2, \pi]), g_n)$ *for some* $\epsilon_2' < \epsilon_2$*,*
(*vi*) $0 < f(r) \leq 1$ *and* $|\dot{f}(r)| \leq 2/\sin(r)$ *for all r, where* $\dot{}$ *means* d/dr*.*

Proof This is similar to Lemma 3.1 of [8]. For (ii) note that $h_{g_n} = 0,\ g' = (f^{(1-n)/2})^{4/(n-1)} g_0$, so the transformation law for the mean curvature gives

$$h_{g'} = \frac{2}{n-2} f^{n(n-1)/2(n-2)} \frac{\partial}{\partial \eta} f^{(1-n)/2}.$$

Since f is a function that depends only on r, the normal derivative $\frac{\partial}{\partial \eta} f^{(1-n)/2} = 0$ by Gauss' Lemma. This way $h_{g'} = 0$ as desired.
The scalar curvature of g' is given by

$$\frac{1}{n(n-1)} R_{g'} = \frac{2}{n} f(\ddot{f} + (n-1) f \cot r) - \dot{f}^2 + f^2.$$

We make the following change of variables:

$$\cos r = \tanh t, \;\; 0 < r < \pi, \;\; \infty > t > -\infty. \tag{14}$$

Put

$$u(t) = f(r(t)) \cosh t. \tag{15}$$

Then $g' = u^{-2}(dt^2 + g_n)$ and

$$\frac{1}{n(n-1)} R_{g'} = \frac{2}{n} u u'' - (u')^2 + \frac{n-2}{n} u^2, \tag{16}$$

where $' = d/dt = -(\sin r) d/dr$. We put

$$B(t) = u^{-n}((u')^2 - u^2 + 1), \tag{17}$$

so that from (16) we get

$$B'(t) = (u^{-n})'(1 - \frac{R_{g'}}{n(n-1)}). \tag{18}$$

We fix $t_0 > \max\{0, \log \cot(\epsilon_2/2)\}$, and set

$$u(t) = \cosh t \;\; \text{for } t \in (-\infty, t_0], \tag{19}$$

hence, $B(t) \equiv 0$ for $t < t_0$. We want to consider the solution u of (17) with a suitable $B(t)$. First note that

$$u(t) \leq \cosh t \;\; \text{for } t \geq t_0, \quad \text{if } B(t) \leq 0 \;\; \text{for} \;\; t \geq t_0, \tag{20}$$

which follows from a comparison argument. Let

$$\begin{cases} B(t) = 0 & \text{for} \;\; t \leq t_0 \\ B(t) \leq 0, -2\delta \leq B'(t) \leq 0 & \text{for} \;\; t_0 \leq t \leq t_0 + 1 \\ B(t) = -\delta & \text{for} \;\; t_0 + 1 \leq t \leq t_1. \end{cases}$$

If δ is sufficiently small, then (17) together with (19) is solvable for u in the interval $(-\infty, t_1)$ with arbitrary $t_1 \geq t_0 + 1$, and $u'(t) > 0$, $u''(t) > 0$ for $t_0 \leq t \leq t_0 + 1$. Therefore, taking $\delta > 0$ smaller we have from (18)

$$|R_{g'} - n(n-1)| \leq 2(n-1) \frac{\cosh^{n+1}(t_0 + 1)}{\sinh t_0} \delta < \epsilon_1, \quad \text{for } t \leq t_0 + 1.$$

We choose t_1 so that $u'(t_1) = 0$ and $u'(t) > 0$ for $t_0 + 1 < t < t_1$. This way $R_{g'} = n(n-1)$ for $t_0 + 1 \leq t \leq t_1$. For $t \geq t_1$ we put $B(t) = B(2t_1 - t)$. Then

$$u(t) = u(2t_1 - t) \;\; \text{for } t \geq t_1. \tag{21}$$

Thus $|R_{g'} - n(n-1)| < \epsilon_1$ for all t, and (v) follows from (19) and (21) via (14) and (15). As for the volume we have

$$Vol(S^n_+, g') = Vol(S^n_+, g_n) + \int_1^{u(t_1)} \frac{du}{u^n u'}$$

and we can see by a long and elementary calculation that the second term of the right hand side converges to $Vol(S^n_+, g_n)$ as $\delta \to 0$. Therefore, (iii) holds for δ small enough, and (iv) follows from an analogous computation. From (20), $u(t) \leq \cosh t$ and hence $|u'(t)| \leq |\sinh t|$ from (17), because $B(t) \leq 0$, which proves (vi). □

Lemma A.10 *Let $(M^n_1, g_1), (M^n_2, g_2), n \geq 3$ be two compact Riemannian manifolds with boundary and $\epsilon > 0$.*

$(\lambda = 1)$ If $h_{g_i} \geq 0,\ R_{g_i}(p_i) = n(n-1), i = 1, 2$ at some points $p_i \in \partial M_i$, then there exists a metric g on $M_1\#M_2$ so that:

- *$(1.a)$ $h_g \geq 0$ and $\left|Vol(M_1\#M_2, g) - \sum_{i=1}^2 Vol(M_i, g_i)\right| < \epsilon$.*
- *$(1.b)$ There are isometric embeddings $\phi_i : (M_i - B_i, g_i) \to (M_1\#M_2, g), i = 1, 2$, where B_i are small balls around $p_i \in \partial M_i$, and $|R_g(x) - n(n-1)| < \epsilon$ for $x \in M_1\#M_2 - (Im(\phi_1) \cup Im(\phi_2))$.*

$(\lambda = 1)$ If $R_{g_i} \geq 0,\ R_{g_i}(p_i) = n(n-1),\ h_{g_i} > 0$ at some points $p_i \in \partial M_i, i = 1, 2$ then there exists a metric g on $M_1\#M_2$ so that:

- *$(0.a)$ $R_g \geq 0$ and $\left|Area(\partial(M_1\#M_2), g) - \sum_{i=1}^2 Area(\partial M_i, g_i)\right| < \epsilon$.*
- *$(0.b)$ There are isometric embeddings $\phi_i : (M_i - B_i, g_i) \to (M_1\#M_2, g), i = 1, 2$, where B_i are small balls around $p_i \in \partial M_i$, and $|h_g(x)| < \epsilon$ for $x \in \partial(M_1\#M_2) - \partial(Im(\phi_1) \cup Im(\phi_2)))$.*

Proof Proof of $\lambda = 1$. By Lemmas A.4, A.7, and A.9 we can take a metric $\bar{g}_i$ of M_i that coincides with g_i outside a small (half) ball B_i containing p_i and such that $|R_{g_i}(x) - n(n-1)| < \epsilon$ for $x \in B_i$, $|Vol(M_i, \bar{g}_i) - Vol(M_i, g_i)| \leq \epsilon/4$ and B_i contains a smaller ball B'_i such that $(B'_i, \bar{g}_i)$ is isometric to a geodesic δ-(half) ball in the unit hemisphere and $Vol(B'_i, \bar{g}_i) < \epsilon/4$.

From Lemma A.9 with ϵ_1, ϵ_2 smaller than ϵ and δ, respectively, we have a small piece $(A[\delta', \delta], g')$ for some $\delta' < \delta$ such that $|R_{g'} - n(n-1)| < \epsilon,\ Vol(A[\delta', \delta], g') < \epsilon/4$ and a neighborhood of each of the boundary components is isometric to a neighborhood of the boundary of the δ-ball inside the hemisphere. This way, the manifold $(M_1\#M_2, g)$ is obtained by putting together $(M - B'_i, \bar{g}_1)$, $(M_2 - B'_2, \bar{g}_2)$ and $(A([\delta', \delta]), g')$.

The proof of $\lambda = 0$ is similar: take $\bar{g}_i$ as above with $|R_{g_i}(x) - n(n-1)| < \epsilon$ for $x \in B_i$, $|Area(\partial M_i, \bar{g}_i) - Area(\partial M_i, g_i)| \leq \epsilon/4$ and B_i contains a smaller ball B'_i such that $(B'_i, \bar{g}_i)$ is isometric to a geodesic δ-(half) ball in the unit hemisphere and $Area((\partial M_i) \cap B'_i, \bar{g}_i) \leq \epsilon/4$. From Lemma A.9 with ϵ_1, ϵ_2 smaller than ϵ and δ, respectively, we have a small piece $(A[\delta', \delta], g')$ for some $\delta' < \delta$ such that $|R_{g'} - n(n-1)| < \epsilon,\ Area(\partial A[\delta', \delta], g') < \epsilon/4$ and a neighborhood of each of the boundary components is isometric to a neighborhood of the boundary of the δ-ball inside the hemisphere. The manifold $(M_1\#M_2, g)$ is obtained by putting together $(M - B'_i, \bar{g}_1)$, $(M_2 - B'_2, \bar{g}_2)$ and $(A([\delta', \delta]), g')$. □

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