EXAM II - SOLUTIONS

(1.) $y''+2y'-3y=2e^t$. Char eqn: $r^2+2r-3=0$

$(r+3)(r-1)=0 \Rightarrow \boxed{r=-3,1}$

$$\boxed{y_H = c_1e^{-3t}+c_2e^t}$$

RHS of eqn: $2e^t \rightarrow$ RESONANCE! try $\boxed{y_p = Ate^t}$

$y_p' = tAe^t + Ae^t$; $y_p'' = tAe^t + 2Ae^t$

Back into eqn: $(\cancel{tAe^t}+2Ae^t)+2(\cancel{tAe^t}+Ae^t)-3\cancel{tAe^t}=2e^t$

get $4Ae^t = 2e^t \Rightarrow \boxed{A=\frac{1}{2}}$

This way: $\boxed{y = y_H + y_p = c_1e^{-3t}+c_2e^t+\frac{1}{2}te^t}$

(2) (a) $t^2y''+2ty'-6y=0$. Char eqn: $r^2+(b-a)r+c=0$

$r^2+r-6=0$

$(r+3)(r-2)=0 \Rightarrow \boxed{r=-3,2}$

$$\boxed{y_H = c_1t^{-3}+c_2t^2}$$

(b) 1st Put the eqn in standard form:

$$\boxed{y''+\frac{2}{t}y'-\frac{6}{t^2}y=\underset{g}{1}}$$

Use variation of parameters: $y_p = v_1y_1+v_2y_2$

Need: $W[y_1, y_2] = y_1 y_2' - y_1' y_2 = t^{-3} \cdot (2t)$

$- (-3t^{-4}) t^2$

$= 2t^{-2} + 3t^{-2} = 5t^{-2}$.

this way:

$$v_1 = \int \frac{-g y_2}{W[y_1, y_2]} dt = \int \frac{-1 \cdot t^2}{W}$$

$$= \int \frac{-t^2}{5t^{-2}} = \int -\frac{1}{5} t^4 = \boxed{-\frac{1}{25} t^5}$$

$$v_2 = \int \frac{g y_1}{W[y_1, y_2]} = \int \frac{1 \cdot t^{-3}}{5 t^{-2}} = \int \frac{1}{5} t^{-1}$$

$$\boxed{= \frac{1}{5} \ln t}$$

we get $y = y_H + y_p = c_1 t^{-3} + c_2 t^2 + \underbrace{-\frac{1}{25} t^5 \cdot t^{-3}}_{v_1 \cdot y_1}$

$$+ \underbrace{\frac{1}{5} \ln t \cdot t^2}_{v_2 y_2}$$

$$\boxed{y = c_1 t^{-3} + c_2 t^2 - \frac{1}{25} t^2 + \frac{1}{5} t^2 \ln t}$$

(3) $y'' + by' = 0$, $y(0) = 2$, $y'(0) = -b$.

char eqn: $r^2 + br = 0$, $r(r+b) = 0$

roots: $\boxed{r = 0, -b}$

Case 1: $\boxed{b=0}$ $\Rightarrow$ $y = c_1 e^{0.t} + c_2 t e^{0.t}$

$$\boxed{y = c_1 + c_2 t}$$

I.C. $y(0) = c_1 = 2$
$y'(0) = c_2 = -b = 0$ $\Big\}$ $\boxed{y = 2}$ NOT possible $y \to 1$.

Case 2: $\boxed{b \neq 0}$

$$\boxed{y = c_1 + c_2 e^{-bt}}$$

$y(0) = c_1 + c_2 = 2$
$y'(0) = -c_2 b = -b$ $\Big\}$ $c_2 = 1$, $c_1 = 1$

$$\therefore \boxed{y = 1 + e^{-bt}}$$

Now if $b > 0$, $y \to 1$ as $t \to \infty$ (b/c $e^{-bt} \searrow 0$)
if $b < 0$, $y \to +\infty$ as $t \to \infty$ (b/c $e^{-bt} \uparrow \infty$).

$\therefore$ $\boxed{\boxed{b > 0}}$ does the trick.

(4) Equation is

$$t^2y'' - ty' + y = 0, \quad t > 0$$

Given: $y = t$ solves the eqn.

<u>Reduction of order:</u> 1st rewrite eqn in std form:

$$y'' \underbrace{- \frac{1}{t}}_{p} y' + \underbrace{\frac{1}{t^2}}_{q} y = 0, \quad t > 0$$

<u>Then,</u>

$$y_2 = t \cdot \int \frac{e^{-\int -\frac{1}{t} dt}}{t^2} dt$$

$$= t \cdot \int \frac{e^{\ln t}}{t^2} dt = t \cdot \int \frac{t}{t^2} dt$$

$$= t \cdot \int \frac{1}{t} dt = t \ln t.$$

therefore $\boxed{y_H = c_1 t + c_2 t \ln t}$

<u>Obs.</u> Here you may have also used that the eqn is a Cauchy-Euler & get gen soln that way.

(5) Assume $y'' = \frac{1}{y}$, $y(0) = 0$
$y'(0) = 0$.

Energy integral lemma says $\frac{1}{2}(y')^2 - F(y) = K$.
In this case, $F(y) = \ln(y)$, so E-L says

$$\boxed{\frac{1}{2}(y'(t))^2 - \ln(y(t)) = K}$$

take $\lim_{t \to 0}$ in the above eqn. We get

$$\underbrace{\frac{1}{2}(0)}_{=0} - \underbrace{\ln(0)}_{\to -\infty} = \underbrace{K}_{K}$$

it is <u>NOT POSSIBLE</u> that $0 + \infty = K$
(= constant)

hence No such solution $y(t)$ exists.