Research Statement

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1. Overview

My research has been primarily in Geometric Analysis. Recently I also became interested to develop and apply computational techniques inspired by Geometry and Topology in Data Analysis problems. In this statement I describe some of my main achievements within Geometric Analysis and my ongoing and future projects in the area. Then I discuss some preliminary results I have obtained applying Geometric and Topological Methods in Data Analysis, as well as my current research projects related to this topic.

2. Geometric Analysis

2.1. **Free-boundary Minimal Surfaces with Morse Index.** In their fundamental paper [28] Schoen and Yau study incompressible surfaces inside 3-manifolds with nonnegative scalar curvature and deduce topological obstructions for an ambient space that contains them. Fischer-Colbrie [10] later studied a parallel situation in which topological obstructions are also found, but the stable minimal surfaces of Schoen-Yau are replaced by *unstable* minimal surfaces (with finite Morse index). Minimal surface theory is a broad and rich area that has seen much activity since.

A trend that resembles the above situation has recently emerged, but now in the world of rigidity theorems for minimal surfaces. First, several rigidity statements were obtained assuming the existence of an *area-minimizing* surface of some kind [3, 4, 6, 8, 27]. Then, Marques and Neves proved in [23] a rigidity result for *unstable* minimal surfaces on the 3-sphere. They showed that any metric on S^3 with positive Ricci curvature and scalar curvature bounded below by 6 always contains an embedded minimal 2-sphere of index one that realizes the *width*[1], and whose area is bounded above by 4π. Furthermore, it asserts that the area of the 2-sphere equals 4π if and only if the metric is the round metric of the 3-sphere, in which case the minimal 2-sphere is equatorial.

In an ongoing collaboration with D. Maximo and I. Nunes we are working on proving an analogue to the Marques-Neves theorem but for manifolds with boundary. In this context, "minimal surface" is replaced by "free-boundary minimal surface." These are defined as follows. Let M^3 be a compact manifold with boundary and $\Sigma^2 \subset M^3$ be an embedded surface *with boundary* so that $\partial\Sigma \subset \partial M$. Consider a smooth variation of Σ by surfaces whose boundary lies in the boundary of M. This is, a smooth family of embedded surfaces with boundary $\{\Sigma_t\}_t$ so that $\Sigma_0 = \Sigma$, and $\partial\Sigma_t \subset \partial M$. A standard calculation gives

$$\frac{d}{dt}\bigg|_{t=0} \text{Area}(\Sigma_t) = -\int_\Sigma \langle \vec{H}, X\rangle dA + \int_{\partial\Sigma} \langle X, \nu\rangle ds,$$

where ν is the outward unit normal of $\partial\Sigma$ tangent to Σ, $\vec{H}$ is the mean curvature vector of Σ in M^3, and X is the variation field associated to the smooth family $\{\Sigma_t\}$, which lives in the tangent space of ∂M since the surfaces of the family all have boundary lying in ∂M. From this, it follows that Σ is a critical point for the area functional over all such possible variations only if $\vec{H} = 0$ and $\nu \perp \partial M$. If this is the case we say that Σ is a *free-boundary minimal surface.*

After some preliminary calculations, the problem (originally suggested by Marques) takes the following form:

[1] See e.g [23] for a precise definition.

Conjecture 2.1. *Let g be a metric on the 3-ball B with nonnegative Ricci curvature and so that $h \geq g|_{\partial B}$, where h is the second fundamental form of the boundary with respect to the inward-pointing normal. Then, there exists an embedded, free-boundary minimal disk $D^2 \subset B^3$ of index one, such that $W(B^3, g) = |D|$ and*

$$L(\partial D) \leq 2\pi. \tag{1}$$

The equality $L(\partial D) = 2\pi$ holds if and only if g is the flat metric of the unit ball, in which case D is a hyperplane intersect the ball.

In the above conjecture $L(\partial D)$ stands for the length of the boundary of D, and $W(B^3, g)$ is the width of (B^3, g), which is a min-max invariant of the metric. (See e.g. [11,23] for more details.)

Our strategy to prove Conjecture 2.1 is divided into two parts. First we prove existence of the free-boundary minimal surface using min-max methods. Then we use an argument involving mean curvature flow to prove the rigidity statement. We have been able to prove several intermediate steps of the project, including the existence of the minimal surface –but we are still working on proving that it has Morse index 1. Once that is proved, we know how to obtain inequality (1). On the other hand, we know how to prove the rigidity statement if we can control the behavior of the second fundamental form of the boundary along its evolution under mean curvature flow. Part of our current efforts are focused on proving those estimates.

Rigidity statements for unstable free-boundary minimal surfaces on manifolds with boundary are interesting but difficult to come by. Related problems to Conjecture 2.1 include rigidity statements for stable free-boundary minimal surfaces on manifolds with boundary such as the solid cylinder $D \times \mathbb{R}$, and so on. An ambitious, long term project is to prove a Willmore-type theorem for minimal catenoids in the ball. This is an active area of research that will keep us busy for a while.

2.2. **Mass-Capacity and Classical Geometric Inequalities using Geometric Flows.** Geometric flows have played a fundamental role in recent developments in differential geometry. One that has attracted considerable interest is the so-called inverse mean curvature flow (IMCF), introduced by Geroch in the 1970's. The flow was originally used to give a heuristic proof of the (Riemannian) Penrose inequality from General Relativity, which states that

$$m \geq \sqrt{A/16\pi}, \tag{2}$$

where m is the ADM-mass (of the universe) and A is the surface-area of all the black holes inside it.

Geroch's argument was based on a monotonicity formula for the Hawking mass along the IMCF, but it lacked a weak PDE theory needed to guarantee that the IMCF exists –and a proof that the Hawking mass can be defined, and remains monotonic, in such weak setting. These gaps were filled by Huisken and Ilmanen in their well known, beautiful paper [17]. There, they develop a weak existence theory for IMCF and eventually are able to prove (a slightly weaker version of) inequality (2): they showed that it holds when A is replaced by the area of any single black hole.

In [18] Huisken and Ilmanen generalized their existence theory for the IMCF to arbitrary dimensions. Unfortunately, though, the main application of the technique –the Penrose inequality– does not follow in arbitrary dimensions by using IMCF since the Geroch monotonicity formula for the Hawking mass is based on the Gauss-Bonnet theorem.

In my joint work with A. Freire we found the first application of the IMCF in arbitrary dimensions. We proved a mass-capacity inequality for conformally flat manifolds with minimal boundary, and a volumetric Penrose inequality –both with rigidity statements.

Theorem 2.2 (Freire-Schwartz [12]). *Given $(M^n, g), n \geq 3$ conformally flat, asymptotically flat with nonnegative scalar curvature and with minimal boundary Σ, we have*

$$m \geq C_g(\Sigma), \tag{3}$$

and equality holds if and only if (M, g) is the Riemannian Schwarzschild manifold. Here, m is the ADM-mass and $C_g(\Sigma)$ the capacity of the boundary.

The volumetric Penrose inequality, which is an improvement to my earlier result in [33], states the following:

Theorem 2.3 (Freire-Schwartz [12]). *For (M, g) as above we have*

$$m \geq 2\,(V_0/\beta_n)^{\frac{n-2}{n}} \tag{4}$$

with equality holding if and only if (M, g) is the Riemannian Schwarzschild manifold. Here, m is the ADM-mass, V_0 is the Euclidean volume of the region that Σ bounds, and β_n the volume of the Euclidean unit n-ball.

The proofs of inequalities (3) and (4) use a generalization of classic Pólya-Szegö and Aleksandrov-Fenchel inequalities for surfaces in $\mathbb{R}^3$ which Freire and I were able to extend to arbitrary dimensions (with rigidity) using IMCF [12]. They are of independent interest, and stated as follows.

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain –not necessarily connected– with mean-convex boundary $\Sigma = \partial\Omega$ which is *outer-minimizing* in Ω^c. (This is, for $\Omega' \supset \Omega$, the area $|\partial\Omega'|$ is no less than $|\Sigma|$.) If we let $H_0 > 0$ be the mean curvature of Σ, $C_0(\Sigma)$ its capacity, and ω_{n-1} is the volume of the round $(n-1)$-sphere in $\mathbb{R}^n$, the generalized Pólya-Szegö inequality states that:

Theorem 2.4 (Freire-Schwartz [12]). *Let Σ be as above. Then*

$$C_0(\Sigma) \leq \frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0, \tag{5}$$

with equality if and only if Ω is a single round n-ball.

The generalized Aleksandrov-Fenchel inequality is:

Theorem 2.5 (Freire-Schwartz [12]). *Let Σ be as above. Then*

$$(A_0/\omega_{n-1})^{\frac{n-2}{n-1}} \leq \frac{1}{(n-1)\omega_{n-1}} \int_\Sigma H_0 d\sigma_0, \tag{6}$$

with if and only if Ω is a single round n-ball.

The mass-capacity inequality (3) can potentially be used to prove the Penrose inequality in arbitrary dimensions, and that is one direction Freire and I are exploring. Our belief is based on the fact that the only part of Bray's proof of the Penrose inequality [2] (and its subsequent generalization [5]) which depends on the Positive Mass Theorem is where it is used to prove the mass-capacity inequality. Another interesting, related open problem that we are considering is the Penrose inequality for conformally flat manifolds, in arbitrary dimensions.

2.3. **Black Rings, the Shrink-Wrap Principle, and Mean Curvature Flow.** An exciting development in the General Relativity community was the recent discovery of non-spherical black holes. In their work [9], the physicists Emparan and Reall constructed "black rings," which are (4+1)-dimensional black hole spacetimes having black hole topology $S^1 \times S^2$. A natural question that arises is to determine what is the Riemannian analogue of this result.

Two lines of works are related to the issue. First, there is a theorem by Meeks-Simon-Yau [24, 25] which states that (a) the outermost minimal surface of an asymptotically flat 3-dimensional Riemannian manifold with nonnegative scalar curvature is, topologically, a 2-sphere, and that (b) the complement of the region enclosed by the outermost minimal surface is diffeomorphic to $\mathbb{R}^3$ minus a finite number of balls. Then, there is the Galloway-Schoen theorem [13, 14], where a generalization of (a) is proved by showing that in dimensions three and above, the outermost minimal hypersurface of an AF manifold with nonnegative scalar curvature must be of positive Yamabe type (i.e. it admits a metric of positive scalar curvature). In view of this, one could argue that the Riemannian version of Emparan and Reall's construction should be proving the existence of high-dimensional AF manifolds with non-negative scalar curvature that contain outermost minimal hypersurfaces with non-spherical topology. This is exactly what I addressed in [30]. I was able to prove the following:

Theorem 2.6 (Schwartz [30])**.** *For any $n, m \geq 1$ there exists an asymptotically flat, scalar flat $(n + m + 2)$-dimensional Riemannian manifold (M, g) with outermost apparent horizon which is an outermost smooth minimal hypersurface with topology $S^n \times S^{m+1}$.*

The techniques involved in the proof of this result are unrelated to the ones used by Emparan and Reall. The main difficulty I had to overcome to prove the above Theorem was to be able to show that the minimal hypersurfaces I constructed were *outermost.* Generally speaking, the issue of being outermost is not well understood. But there are some heuristic arguments that may help clarify the situation. I am particularly interested in the so-called Shrink-Wrap Principle, which states the following:

Conjecture 2.7 (Shrink-wrap Principle)**.** *The evolution of a celestial sphere*[2] *under mean curvature flow inside an asymptotically flat manifold with nonnegative scalar curvature converges to the outermost minimal hypersurface in it.*

If, as expected, Conjecture 2.7 is true, then we have a way to connecting the evolution under mean curvature flow of a large sphere –and the singularities that may develop along it– to the outermost minimal hypersurface in the ambient space. Consequently, if we understand the formation of singularities along this flow we could, in principle, find new topological obstructions and obtain a better understanding of outermost minimal hypersurfaces.

In an ongoing collaboration with E. Cabezas-Rivas we are trying to make inroads towards proving Conjecture 2.7. Our first step consists in studying the formation of singularities along MCF, with particular focus on singularities that *increase* the topological complexity of the flowing hypersurface.

Conjecture 2.8. *For $n \geq 3$ and $1 \leq k \leq n-1$, there exits a weak solution (in the sense of* [20]*) of the mean curvature flow, with initial condition an almost-round sphere, which is smooth for all times $t \geq 0$ except only at one singular time $t_0 > 0$, and so that the topology of the hypersurfaces remains spherical for $0 \leq t < t_0$, but becomes a product of spheres $S^k \times S^{n-k}$ after the singular time $t > t_0$. Furthermore, the flow converges to a smooth hypersurface of topology $S^k \times S^{n-k}$ as $t \to \infty$.*

[2] I.e. a large enough coordinate sphere.

Conjecture 2.8 is of independent interest since no examples of solutions to MCF that increase topological complexity are known to exist (see [35, 36]). Furthermore, such behavior goes against the general expectation that MCF is qualitatively similar to Ricci flow and thus only simplifies the topology of the evolving hypersurface (cf. [16]).

The particular setting where we expect this phenomenon to occur is when we run the MCF starting at a celestial sphere inside one of the manifolds of Theorem 2.6. The belief is based on the heuristic argument that if Conjecture 2.7 holds, then celestial spheres flowing under MCF converge to the outermost minimal hypersurface in the ambient space –and this is a topological ring if we are inside one of the manifolds of constructed in Theorem 2.6. So far, we have been able to prove several intermediate steps of Conjecture 2.8. The arguments go along the lines of the works in [1, 7, 34].

2.4. **Yamabe Problem and Topological Invariants.** The Yamabe problem on a Riemannian manifold consist in finding conformally-related metrics having constant scalar curvature. The problem is equivalent to solving a nonlinear PDE with critical exponent. The Yamabe problem was solved, for closed manifolds, in the well-known works of Yamabe, Aubin and Schoen. On a manifold with boundary a similar problem can be posed, and a boundary condition for the PDE is added. The geometric meaning of this condition is that the boundary has now been prescribed a function as its mean curvature. In my PhD dissertation I proved that given a noncompact, scalar flat manifold with super-linear volume growth at infinity with positive mean curvature on its boundary, and *any* smooth function f on its boundary, there exists a conformally related, scalar-flat metric with prescribed mean curvature f on the boundary. The result is unexpected since, in the compact case, the prescribed mean curvature function f must integrate to zero on the boundary. The (slightly more general) PDE statement is the following:

Theorem 2.9 (Schwartz [29]). *Let (M, g) be a noncompact Riemannian manifold as above. Let f be a smooth function on ∂M and $\beta > 1$. There exists smooth function $u > 0$ on M with (appropriate growth at infinity) so that*

$$\begin{cases} \left(\Delta_g - \frac{n-2}{4(n-1)} R(g)\right) u &= 0 \qquad \text{in } M, \\ \left(\frac{\partial}{\partial \eta} + \frac{n-2}{2} h(g)\right) u &= \frac{n-2}{2} f u^{\beta} \quad \text{on } \partial M. \end{cases}$$

Here, $R(g)$ is the scalar curvature of g, $h(g)$ is the mean curvature of ∂M with respect to g, and $\partial/\partial\eta$ is the outward-pointing normal on the boundary.

In particular, when β is the critical exponent $\beta = n/(n-2)$, the conformally related metric $\tilde{g} = u^{4/(n-2)} g$ defines a complete, scalar flat metric on M with mean curvature f on ∂M.

A related geometric quantity is the so-called Yamabe invariant, which is an invariant of the smooth structure of a manifold. It is defined as follows. First we define the Yamabe *constant* of a Riemannian manifold (M, g) –which is an invariant of the conformal structure only– as the infimum of the total scalar curvature among unit-volume, conformally-related metrics, and we denote it by $Y(M, [g])$. The Yamabe invariant of M is then defined as the supremum of the Yamabe constant among all of its conformal classes of metrics, and we shall call it $\sigma(M)$.

Very few Yamabe invariants have been computed. The main difficulty for doing so lies on the fact that it is a min-max quantity. In [33] I found upper bounds for the Yamabe constant of outermost minimal hypersurfaces inside manifolds that satisfy the Penrose inequality. The statement is the following:

Theorem 2.10 (Schwartz [32]). *Consider an asymptotically flat manifold $(M^n, g), n \geq 3$ with nonnegative scalar curvature containing an outermost minimal hypersurface Σ.*

(i) *If $n = 3$, then $\#\{$components of $\Sigma\} \leq \frac{1}{2\pi} m^2 \omega_2 (||R^M||_{L^\infty(\Sigma)} + 2||Ric(\nu)||_{L^\infty(\Sigma)})$. Equality is attained above if and only if the part of (M^3, g) outside the outermost minimal hypersurface is isometric to the Riemannian Schwarzschild 3-manifold of mass m.*

(ii) *If (M^n, g) satisfies the Penrose inequality (e.g. if $4 \leq n \leq 7$), then*

$$Y(\Sigma, [g|_\Sigma]) \leq (2m)^{\frac{2}{n-2}} (\omega_{n-1})^{\frac{2}{n-1}} (||R^M||_{L^\infty(\Sigma)} + 2||Ric(\nu)||_{L^\infty(\Sigma)}).$$

Equality is attained above if and only if the part of (M^n, g) outside the outermost minimal hypersurface is isometric to the Riemannian Schwarzschild n-manifold of mass m.

In my work [31] I proved that the Yamabe invariant for manifolds with boundary (which is defined in a similar way to the closed-manifold case) satisfies monotonicity properties with respect to connected-sum along the boundary. The precise statement is:

Theorem 2.11 (Schwartz [31]). *Let M_1, M_2 be smooth n-manifolds with boundary, $n \geq 3$. Let $M_1 \# M_2$, $M_1 \sqcup M_2$ denote their connected sum along the boundary and disjoint union, respectively. Then $\sigma(M_1 \# M_2) \geq \sigma(M_1 \sqcup M_2)$, where*

$$\sigma(M_1 \sqcup M_2) = \begin{cases} -(|\sigma(M_1)|^{\frac{n}{2}} + |\sigma(M_2)|^{\frac{n}{2}})^{\frac{2}{n}} & \sigma(M_1), \sigma(M_2) \leq 0, \\ \min\{\sigma(M_1), \sigma(M_2)\} & \textit{otherwise.} \end{cases}$$

The above theorem is the manifold-with-boundary analogue of a classic theorem of Kobayashi for the monotonicity of the Yamabe invariant under connected sum in the closed-manifold case. The result implies, in particular, that the numerical value of the Yamabe invariant of handlebodies is maximal, equal to the one of the n-ball.

3. Geometric and Topological Methods in Data Science

> *"A new postulate is accepted now in biosciences: the information provided by the data in huge volumes without prior hypothesis is complementary and sometimes necessary to conventional approaches based on experimentation. In the massive approaches it is the formulation of a relevant hypothesis to explain the data that is the limiting factor. The search logic is reversed and the limits of induction [need] to be considered."*

The above paragraph, extracted from Wikipedia's page on big data, depicts the undergoing revolution in biosciences. The paradigm shift that the big data era brings will most likely penetrate all scientific disciplines. As massive data sets become widely available, we are no longer bound –in principle– to fit the data within preexisting models. On the contrary, data should be allowed to "speak freely." How this is done, using computational methodologies based on mathematically-sound techniques, is my main source of inspiration in this area. Particularly, I am interested in finding ways to tackle the central data analysis problem of *Knowledge Discovery*, which consists in extracting information from large data sets about local and global structures and their interactions, and then transforming this output into an understandable format for further use. The matter becomes increasingly relevant in an age where standard data-mining methodologies are rendered obsolete.

I have recently started several collaborations in which I apply and develop new techniques inspired by geometric and topological methods to data science problems. Some preliminary results have been obtained, but much work is still needed. In what follows I list some problems I am currently working on and describe my future plans in the subject.

3.1. **Topology Backs Holistic Medicine Claim.** (Joint with CSURE REU student Louis Xiang, and Kwai Wong from NICS.) The holistic concept in alternative medical practice upholds that "all of people's needs should be taken into account." In other words, the body is seen as a whole. The holistic point of view can be scientifically validated by determining whether different bodily variables are related to one another. In my recent joint work we provide strong evidence supporting this claim. More precisely, we establish the existence of at least one fully non-linear relationship involving two bodily variables.

To "prove" this, we consider the ICU dataset from the PhysioNet Challenge (see §3.3 below). We look at the set consisting of all the measurements of the eight most populated variables: Systolic Pressure; Diastolic Pressure; Heart Rate; Mean Arterial Pressure; Urine; Temperature; Respiratory Rate; and GCS. The data set consists of about 300,000 points in eight-dimensional Euclidean space. We used topology-based algorithms such as *javaPlex* –modified to run in parallel– and were able to compute the first three Betti numbers for the *underlying space* of the dataset. This is, the Betti numbers of the space we assumed the points were sampled from. With the help of the Darter supercomputer at ORNL, we were able to show that the underlying space has Betti numbers $(b_0, b_1, b_2) = (1, 2, 0)$. The simplest topology with this sequence of Betti numbers is a space that deformation-retracts to a figure eight, or a wedge of two circles. In other words, the eight most commonly measured variables in the ICU dataset are shaped in what resembles a noisy, thick figure eight. The simplest possible explanation for this phenomenon is that there is one nonlinear equation that two of the eight variables satisfy. The next step in this project consists in finding the nonlinear relation(s), and then showing that they hold with a high degree of confidence.

3.2. **Topology-Based Classifiers.** In a joint work with Erik Ferragut (ORNL) we are building a classifier based on topological techniques such as *Mapper* and *javaPlex*. Theirs will be the first classifying algorithm of the kind. (Known algorithms for dealing with the classification problem include support vector machines and k-nearest neighbor, among others.) The motivation for the joint project is based on the fact that topological techniques have proved successful for finding data structures that were previously unknown, as in [26]. This fuels the expectation that classification algorithms based on these techniques will be very powerful, and particularly significant for dealing with high-dimensional data analysis. Potential applications include tackling some of the hardest problems in cyber security related to the discovery of anomalies.

3.3. **ICU Patient Survival Prediction.** The 2012 PhysioNet.org Computing in Cardiology Challenge released a data set containing records from twelve-thousand ICU stays of adult patients who were admitted to cardiac, medical, surgical, and trauma ICUs for a wide variety of reasons. Up to forty-two variables were recorded at least once during the first forty-eight hours after admission to the ICU. The goal of the Challenge was to predict, with the highest accuracy possible, *patient survival* (i.e. to determine whether patients will have an in-hospital death) by using only the data of the first forty-eight hours spent in the ICU.

In my current joint work with Xiaopeng Zhao (BME) and our student Adam Aaron we apply topological and geometric techniques to obtain patient-survival predictions on this

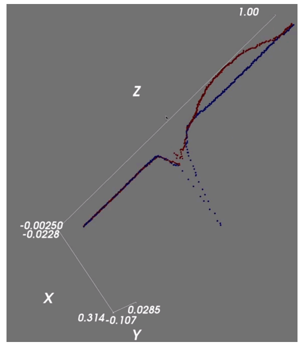

FIGURE 1. MEG Analysis for Face vs. Scrambled Face Shows Differences

dataset. We believe there is room for improvement since the winning team of the 2012 Challenge had a fairly inaccurate prediction –it was right about 50% of the time.

3.4. **Decoding the Brain.** A recent challenge within the Kaggle.com website consists in predicting visual stimuli from human brain activity recordings. The experimental data is obtained by measuring the concurrent magnetoencephalography (MEG) recordings of human brain activity when exposed to two different visual stimulus: picture of a face, and a scrambled picture of a face. When a subject is presented a stimulus, the relation between the pattern of recorded signal and the category of the stimulus may provide insights on the underlying mental process. Among the approaches to analyze the relation between brain activity and stimuli, the one based on predicting the stimulus from the concurrent brain recording is called *brain decoding*.

I am currently working on this problem together with Hairong Qi (EECS) and our joint graduate student Cristian Capdevila. Encouraging preliminary results have been obtained using topological methods in conjunction with techniques such as Local Linear Embedding; a depiction of our progress is in Figure 1, where averaged brain-activity behavior is portrayed for a fixed individual. The two curves (red and blue) each represent brain activity states as time (z-axis) progresses: red curve depicts average activity for "scrambled face"; blue curve represents "face" brain activity. It is worth noticing that the face/scrambled face picture is shown to each individual after one-half second; this explains why both red and blue curves match for initial values of the z-parameter (time).

3.5. **Conformal Geometry and Morphometrics.** Morphometrics refers to the quantitative analysis of form or shape. Morphometric analyses are commonly performed on organisms. For example, in Evolutionary Biology they are useful for analyzing fossil records, impact of mutations on shape, developmental changes in form, as well as correlations between ecological factors and shape. Morphometrics can be used to quantify a trait of evolutionary significance; detecting changes in the shape can also be used to determining function or evolutionary relationships.

A standard procedure to quantify the similarity or dissimilarity of shapes is by using the so-called *procrustes distance*, which is computed as follows. Each shape S, S' is usually represented by a fixed amount of "landmark points;" these are points $\{p_i\}, \{p'_i\}$ in S, S' respectively, chosen to be homologous and as far as possible from each other. The procrustes distance between the shapes is the infimum over the set of all Euclidean transformations[3] E of the quantities $\sum_i ||E(p_i) - p_i'||^2$.

Together with my PhD student J. Mike we are working on extending the results obtained by Lipman, Al-Aifari and Daubechies [22] regarding the use of conformal geometry for finding a continuous analogue of the procrustes distance from above. Applications of these types of techniques are of particular interest in Evolutionary Biology and related fields.

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