Example of Poisson formula in action:

This is an example of the Poisson formula in action. We have not seen many examples. The reason is that practical evaluation of the integrals is
(a) easy, but these are often cases where Poisson's formula is not needed; or (b) practically impossible, so the example leaves students wondering how useful the whole thing is; or
(c) in between: then the power of the formula is clearly displayed, but the examples require lengthy calculations.

For (b), I should remark that it is well possible to extract useful information from 'unevaluated' integrals, but many of you will not have seen this done and therefore not have aquired the skill to do it; so such examples would be more distracting than illuminating. — Here is an example for the intermediate case (c); note that Hwk #16 (2nd paragraph) is an example for the type of calculations needed in the practical evaluation of Poisson's formula in those cases where such evaluation is actually possible.

We want to solve $\Delta u = 0$ in the unit disk $|r| < 1$, with the boundary condition $u(1,\varphi) = g(\varphi) := |\sin\varphi|$. The solution formula says

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\varphi')} g(\varphi')\,d\varphi' \,.$$

So the formula gives us:

$$\begin{aligned} u(r,\varphi) &= \frac{1}{2\pi}\int_0^{\pi} \frac{(1-r^2)\sin\varphi'}{1+r^2-2r\cos(\varphi-\varphi')}\,d\varphi' + \frac{1}{2\pi}\int_\pi^{2\pi} \frac{(1-r^2)(-\sin\varphi')}{1+r^2-2r\cos(\varphi-\varphi')}\,d\varphi' = \\ &= \frac{1-r^2}{2\pi}\int_0^{\pi} \frac{\sin\varphi'\,d\varphi'}{1+r^2-2r\cos(\varphi-\varphi')} + \frac{1-r^2}{2\pi}\int_0^{\pi} \frac{\sin\varphi'\,d\varphi'}{1+r^2+2r\cos(\varphi-\varphi')} \,. \end{aligned} \tag{1}$$

The big task is to evaluate this integral. This is, in principle, a Calculus 2 task, but it requires full industrial strength.

It seems enticing to combine the two fractions, harvesting algebraic simplification. However, we will make the substitution $\tan\varphi'/2 = t$ to get rig of the trig functions and have a rational function in t to integrate instead. This rational function is integrated by means of partial fraction decmposition, and we would just have to take apart the fractions again then, if we merged them now.

Instead we note that the second integral arises from the first by changing r into $-r$, and this will save us half the work later.

PRINCIPLE: Whenever you have an integrand that is a rational expression involving $\sin\varphi'$ and $\cos\varphi'$, the substitution $\tan\varphi'/2$ makes the integrand rational. (Variant: Only if this expression in $\sin\varphi'$ and $\cos\varphi'$ actually depends on $\sin^2\varphi'$, $\cos^2\varphi'$ and $\sin\varphi'\cos\varphi'$ alone, only then can we mkae a shortcut and substitute $\tan\varphi' = s$ instead.) Here are all relevant ingredients of the 'tangent-half' substitution:

$$t = \tan\frac{\varphi'}{2}\,, \quad d\varphi' = \frac{2dt}{1+t^2}\,, \quad \cos\varphi' = \frac{1-t^2}{1+t^2}\,, \quad \sin\varphi' = \frac{2t}{1+t^2} \tag{2}$$

$$\begin{aligned} \frac{\sin\varphi'\,d\varphi'}{1+r^2-2r\cos(\varphi-\varphi')} &= \frac{4t\,dt}{(1+t^2)[(1+r^2)(1+t^2)-2r(1-t^2)\cos\varphi-4rt\sin\varphi]} \\ &= \frac{4t\,dt}{(1+t^2)[(1+r^2+2r\cos\varphi)t^2-4tr\sin\varphi+(1+r^2-2r\cos\varphi)]} \end{aligned} \tag{3}$$

We let, for abbreviation,

$$(1+r^2 \pm 2r\cos\varphi) =: C_\pm \quad\text{and}\quad 2r\sin\varphi =: S \quad\text{hence}\quad C_-C_+ - S^2 = (1-r^2)^2$$

and attempt a partial fraction decomposition (PFD)

$$\frac{4t}{(1+t^2)(C_+t^2 - 2tS + C_-)} = \frac{at+b}{t^2+1} + \frac{c(t-S/C_+)+d}{C_+(t-S/C_+)^2 + (C_-C_+ - S^2)/C_+}\ . \tag{4}$$

Multiplying this equation with t^2+1, then taking the limit $t \to i$ produces[1]

$$ai + b = \frac{4i}{-2iS - C_+ + C_-} = \frac{-8S + 4i(C_- - C_+)}{(C_- - C_+)^2 + 4S^2} = \frac{-4r\sin\varphi - 4ir\cos\varphi}{4r^2}$$

So

$$a = -\frac{\cos\varphi}{r} \quad\text{and}\quad b = -\frac{\sin\varphi}{r}\ .$$

To get c and d, we multiply the PFD ansatz (4) by $C_+(t-S/C_+)^2 + (C_-C_+ - S^2)/C_+$ and then let $t \to S/C_+ + i(C_-C_+ - S^2)^{1/2}/C_+$.

$$\begin{aligned}
ci(1-r^2)/C_+ + d &= \frac{4t}{1+t^2}\Big|_{t\to S/C_+ + i(1-r^2)/C_+} = \frac{4SC_+ + 4i(1-r^2)C_+}{C_+^2 + S^2 - (1-r^2)^2 + 2i(1-r^2)S} \\
&= \frac{4C_+\,[S + i(1-r^2)]\,[8r^2 + 4r(1+r^2)\cos\varphi - 2i(1-r^2)S]}{[8r^2 + 4r(1+r^2)\cos\varphi + 2i(1-r^2)S]\,[8r^2 + 4r(1+r^2)\cos\varphi - 2i(1-r^2)S]} \\
&= \frac{16C_+\,[2r\sin\varphi + i(1-r^2)]\,[2r^2 + r(1+r^2)\cos\varphi - ir(1-r^2)\sin\varphi]}{4\,[\,(4r^2 + 2r(1+r^2)\cos\varphi)^2 + 4(1-r^2)^2r^2\sin^2\varphi\,]} \\
&= \frac{4C_+[2r^2(1+r^2)\sin\varphi\,\cos\varphi + r(1+r^2)^2\sin\varphi + 2ir^2(1-r^2)\cos^2\varphi + ir(1-r^2)(1+r^2)\cos\varphi]}{4r^2(1+r^2)^2 + 16r^4\cos^2\varphi + 16r^3(1+r^2)\cos\varphi} \\
&= \frac{C_+\,[r(1+r^2)\sin\varphi\,(2r\cos\varphi + 1 + r^2) + ir(1-r^2)\cos\varphi\,(2r\cos\varphi + 1 + r^2)]}{r^2\,(1+r^2+2r\cos\varphi)^2} \\
&= \frac{(1+r^2)\sin\varphi + i(1-r^2)\cos\varphi}{r}
\end{aligned}$$

Hence

$$c = \frac{C_+}{r}\cos\varphi \quad\text{and}\quad d = \frac{1+r^2}{r}\sin\varphi\ .$$

By (3), this concludes the PFD of the first integral in the formula (1) for u. The second integral arises from the first by replacing r with $-r$ (and therefore S, C_+, C_- with $-S$, C_-, C_+ respectively). This means in particular that the $(at+b)/(t^2+1)$ terms of the two integrals will cancel. We conclude

$$\begin{aligned}
u(r,\varphi) = \frac{1-r^2}{2\pi r}\int_0^\infty \Bigg(&\frac{C_+\cos\varphi\,(t-S/C_+) + (1+r^2)\sin\varphi}{C_+(t-S/C_+)^2 + (C_-C_+ - S^2)/C_+} \\
&- \frac{C_-\cos\varphi\,(t+S/C_-) + (1+r^2)\sin\varphi}{C_-(t+S/C_-)^2 + (C_-C_+ - S^2)/C_-}\Bigg)\,dt
\end{aligned}$$

[1]If you are not familiar with the 'cover-up' or residue method to find a PFD practically, even in the case of complex roots, you may wish to refer to my notes on PFD on the web, linked from my M231 course notes as supplementary material to Laplace transform (b/c that's where PFD's were needed). Of course any other method you may know works as well, but (I bet) far more tediously.

Now notice that $\int \frac{c\tau+d}{\tau^2+e^2}\,d\tau = \frac{c}{2}\ln(\tau^2+e^2) + \frac{d}{e}\arctan(\frac{\tau}{e})$. Therefore

$$u(r,\varphi) = \frac{1-r^2}{2\pi r}\lim_{N\to\infty}\left(\frac{\cos\varphi}{2}\left[\ln\Big(\tau^2+\frac{(1-r^2)^2}{C_+^2}\Big)\right]_{-S/C_+}^{N-S/C_+} + \frac{(1+r^2)\sin\varphi}{1-r^2}\left[\arctan\frac{C_+\tau}{1-r^2}\right]_{-S/C_+}^{N-S/C_+}\right.$$
$$\left. - \frac{\cos\varphi}{2}\left[\ln\Big(\tau^2+\frac{(1-r^2)^2}{C_-^2}\Big)\right]_{S/C_-}^{N+S/C_-} - \frac{(1+r^2)\sin\varphi}{1-r^2}\left[\arctan\frac{C_-\tau}{1-r^2}\right]_{S/C_-}^{N+S/C_-}\right)$$

Since we have split the integral into two integrals ($\int(A-B) = \int A - \int B$), we have explicitly written the upper limit ∞ as a limit $\lim_{N\to\infty}\int^N$. Otherwise the splitting would have resulted in an expression $\infty-\infty$. If you find this mysterious, you can study the same phenomenon when you try to evaluate, e.g., $\int_0^\infty \frac{2t}{(t+1)(t+3)} = \lim_{N\to\infty}(\int_0^N \frac{dt}{t+1} - \int_0^N \frac{dt}{t+3})$.

Back to our integral, we further evaluate

$$u(r,\varphi) = \frac{1-r^2}{2\pi r}\left(\frac{\cos\varphi}{2}\lim_{N\to\infty}\ln\frac{(N-S/C_+)^2(1-r^2)^2/C_+^2}{(N+S/C_-)^2(1-r^2)^2/C_-^2}\right.$$
$$\left. - \frac{\cos\varphi}{2}\ln\frac{(S^2+(1-r^2)^2)C_-^2}{C_+^2(S^2+(1-r^2)^2)} - \frac{1+r^2}{1-r^2}\sin\varphi\left[\arctan\frac{-S}{1-r^2} - \arctan\frac{S}{1-r^2}\right]\right)$$
$$= -\frac{1-r^2}{2\pi r}\cos\varphi\ln\frac{1+r^2-2r\cos\varphi}{1+r^2+2r\cos\varphi} + \frac{1+r^2}{\pi r}\sin\varphi\arctan\frac{2r\sin\varphi}{1-r^2}.$$

After such a lengthy calculation it is wise to check the result against calculational errors. As $r\to 1$, the first term goes to 0, due to the $1-r^2$ coefficient. This is even true when $\cos\varphi = \pm 1$ (which makes the logarithm goe to ∞ as $r\to 1$, because the $(1-r^2)$ coefficients wins against the logarithm. In the second term, the fraction under the arctan goes to $+\infty$ or $-\infty$, depending on whether $\sin\varphi$ is positive or negative. So the whole $\arctan(2r\sin\varphi/(1-r^2))$ goes to $\frac{\pi}{2}\,\mathrm{sign}(\sin\varphi)$ where the signum function sign gives the sign ($+1$, -1, or 0) of its argument. So we indeed verify $u(r\to 1,\varphi) = |\sin\varphi|$. This is already a strong indication that we calculated right, because a random miscalculation would likely have destroyed the boundary condition. Checking that u is indeed harmonic is far more tedious, but can be done in a routine way. It may be preferable to transform into cartesian coordinates first, if this job is done by hand (as opposed to symbolic algebra software).

I should probably point out that Mathematica, one of the major symbolic algebar packages, took 235 sec CPU time (in a new session) to evaluate integral (1), and the result was an output of nearly 2 pages that would take no less skill to bring in a human readable simplified (and manifestly real) form than the evaluation given here by hand.