Homework
UTK – M435 – Partial Differential Equations
Spring 2007, Jochen Denzler, MWF 1:25–2:15, Ayres 205

1. Find all the C^1 functions $u = u(x,y)$ defined on $\mathbb{R}^2$ satisfying the PDE $u_x = 0$. Likewise, find all the C^1 functions $u = u(x,y)$ defined on $\mathbb{R}^2$ satisfying $u_x + u_y = 0$. *Hint: Coordinate transformation; replace one of the independent variables with $z = x - y$.*

2. For $n = 1,2,3,4$, find all homogeneous polynomials $u = u(x,y)$ in 2 variables, of degree n, satisfing $\Delta u = 0$. – Method of undetermined coefficients. *Explanation: A homogeneous polynomial of degree n is a polynomial in which all monomials have the same degree n; e.g., $x^2 - 7xy + 3y^2$ is a homogeneous polynomial of degree 2, but $x^2 + 2xy + 4y^2 - 3x$ is a NON-homogeneous polynomial of degree 2, because the term $-3x$ has lower degree.*

3. For every $n \in \mathbb{N}$ show that $u(r,\varphi) = r^n \cos n\varphi$ and $u(r,\varphi) = r^n \sin n\varphi$ both satisfy $\Delta u = 0$. Rewrite these functions in cartesian coordinates, for $n = 1,2,3,4$.

Use Euler's formula to rewrite the above functions as real and imaginary parts of a certain simple function $f(z)$ of a complex variable z, where $z = x + iy$.

How about negative integers n? Does it still work?

4. Principle: Whenever you have a complex function $f(z)$ that is differentiable, then its real part $u(x,y) = \operatorname{Re} f(z)$ and its imaginary part $v(x,y) = \operatorname{Im} f(z)$ are harmonic: $\Delta u = 0$, $\Delta v = 0$. *Note: The precise definition of differentiability of a complex function would need some qualifications and is discussed in a complex variable course. For our purposes here, you may pretend the following pragmatic 'definition': Whenever a function makes sense for complex variables to plug in, either because the function is given as a power series like e^z, or because it is, e.g., a rational function, and whenever you can find a formula derivative Calc-1 style without running into trouble (like eg., vanishing denominators), then the function is differentiable in the sense of complex variables, too. This is mathematically not quite clean, but good enough for the benefits of big partial understanding to outweigh a puristic approach, which would enforce ignorance on the matter.* — Now find a few harmonic functions (and their domain of definition) from the following complex functions $f_1(z) = \frac{z+1}{z-1}$, $f_2(z) = e^z$, $f_3(z) = \sqrt{z}$ for $(x,y) \in \mathbb{R}^2 \setminus]-\infty, 0] \times \{0\}$. *Note: To express $\sqrt{z}$ as real and imaginary part, solve $(u + iv)^2 = x + iy$ for (u,v) (2 eqns in 2 unknowns).*

Note: This principle works exclusively in 2 dimensions.

5. Find a function u of the special form $u(x,y) = f(x)g(y)$ satisfying $\Delta u = -2u$. Can you find more than one function (other than taking multiples of one and the same function) by the same method?

6. Use the chain rule to calculate $\Delta u = u_{xx} + u_{yy}$ in terms of rotated coordinates (ξ, η) where $\xi = (\cos\alpha)x - (\sin\alpha)y$, $\eta = (\sin\alpha)x + (\cos\alpha)y$ and α is any given angle.

Also calculate the expression u_{xy} in terms of partials with respect to the same variables (ξ, η). *Notice how the Laplacian is invariant under such coordinate changes, whereas the differential operator $\partial^2/\partial x \partial y$ is not.*

7. In cylindrical coordinates in $\mathbb{R}^3$, $x = r\cos\varphi$, $= r\sin\varphi$, z, the Laplacian is $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$. (This follows from basically the same calculation as the one for planar coordinates, you don't need to redo it.)

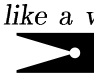
Separate the PDE $\Delta u = 0$ for $u = R(r)Q(\varphi)Z(z)$ into three ODEs and solve two of them (the easy ones). *You would need 431 methods, or otherwise knowledge about Bessel functions, to solve the remaining ODE as well, so we won't go there.*

8. Suppose $u = u(\xi, \eta)$ satisfies $u_{\xi\xi} + u_{\eta\eta} = 0$. Show that $w(x, y) := u(x^2 - y^2, 2xy)$ satisfies $w_{xx} + w_{yy} = 0$.

9. Show: $\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g$.

10. If Ω is a bounded domain with boundary as smooth as you'd like it, show that the Neumann boundary value problem $\Delta u = 0$ in Ω, $\partial_\nu u = g$ on $\partial\Omega$, cannot have a solution unless $\int_{\partial\Omega} g(x)dS(x) = 0$. *Hint: Use the divergence theorem on* $\int_\Omega \Delta u \, d^n x$. (Also note that *if* the problem has a solution u, than $u + C$ is also a solution for every constant C.)

11. Use separation of variables to find solutions to the PDE $\Delta u = 0$ in the wedge $|\varphi| \le \alpha$ that are positive inside the wedge and 0 on the boundary of the wedge. Here α is any angle between 0 and π. What can you say about the radial derivative of u near the vertex of the wedge? How does its size depend on α?

I have a hammer that can also be used for pulling out nails, because the back end of the metal piece is formed like a wedge. If you look closely, it is not an exact wedge but looks rather like this:  . *The little 'hole' does* NOT *serve the purpose to hold a nail. What purpose does it serve? Answer: It reduces mechanical stress inside the wedge. The true equations of elasticity are more complicated than* $\Delta u = 0$, *but the 'philosophy' generalizes: in re-entrant corners, solutions may – and typically will – have large derivatives (and in the example at hand, such derivatives correspond to mechanical stress)*

12. Find the solution to the BVP $\Delta u = 0$ in the unit disk $x^2 + y^2 < 1$, $u = x^2 - xy^2 + y^4$ on the boundary of the unit disk. *Hint: transform all data into polar coordinates, use trig formulas to convert powers of trigs into trigs of multiple angles, and then take the appropriate linear combination of separation solutions. Transform the solution back into cartesian coordinates.*

13. Calculate the Fourier series for the function $f(\varphi) = 1 - 2|\varphi|/\pi$ for $|\varphi| \le \pi$ (and 2π-periodically continued for φ outside this range). Plot in *one* figure the graphs of the sum of the first two, of the first five, and of the first 15, nonvanishing terms in the Fourier series respectively.

14. Same problem, this time with the function $f(x) = \operatorname{sign} x$ for $|x| < \pi$. (By definition, $\operatorname{sign} x$ is 1 or -1 for x positive or negative, respectively.) Here we also want the sum of the first 50 terms in the Fourier series; and we want a graph that zooms in on the interval $|x| < \pi/4$ rather than showing the full interval.

15. We are studying the function

$$P(r,\varphi) = \frac{1}{2\pi} \frac{1-r^2}{1+r^2-2r\cos\varphi}$$

in the unit disc $|r| < 1$. Your first job is to draw the level lines of this function, in order to get an idea how it looks. (If you have technology, use it first to get a good guess. Then prove it analytically. Without technology, proceed to the analytical proof directly.) *Hint: Write $P(r,\varphi) = A/(2\pi)$ in cartesian coordinates, then simplify to the form $(x-?)^2 + y^2 = ?^2$ with appropriate expressions instead of the question marks.*

Also graph a slice of the function along the x axis: $-1 \le x < 1$, $y = 0$; and a slice along the y axis: $x = 0$, $-1 \le y \le 1$.

16. Next graph a few slices along circles in one common figure: $2\pi P(r,\varphi)$ over the interval $\varphi \in [-\pi, \pi]$, for the cases $r \in \{0, 1/5, 1/3, 1/2, 3/5, 3/4\}$. Don't be stingy with space, but draw a nice figure with TLC.

Show that for each $r < 1$, it holds $\int_{-\pi}^{\pi} P(r,\varphi), d\varphi = 1$. *Hint: Before the advent of textbook obesity, it used to be known from Calc 2 that the wise substitution for any* rational *expression of* $\cos\varphi$ *and* $\sin\varphi$ *under the integral is:* $t = \tan(\varphi/2)$. *Once you have faithfully carried out this substitution, you should find that the substitution* $t = \frac{1-r}{1+r} s$ *does a great cleanup of annoying constants.*

17. Show that $\Delta P(r,\varphi) = 0$ in the unit disk $|r| < 1$.

18. Write down the Fourier series for $f(x) = x/2$ on the interval $]-\pi, \pi[$. What does the Parseval identity say, explicitly, about a certain infinite sum of numbers? Write out the neat formula thus obtained.

Assuming the theorem that asserts convergence of the series to the function for piecewise C^1 functions at every point of continuity (in other words, in this example, the series converges to $f(x)$ for every x other than odd multiples of π), write out explicitly the formulas obtained from choosing $x = \pi/2$, $x = \pi/3$ and $x = \pi/4$ respectively.

19. Write down the Fourier series for $f(x) = x^2$ on the interval $[-\pi, \pi]$. What formula does the Parseval identity give us? What formulas do we obtain from choosing $x = \pi/2$, $x = \pi/3$ and $x = \pi/4$ respectively?

20. Write down the sine Fourier series of $f(x) = x(\pi - x)$ on the interval $[0, \pi]$. We will now use it to solve the Boundary Value Problem $\Delta v = 0$ in the square $]0, \pi[\times]0, \pi[$ subject to the BCs $v(x, 0) = 0$, $v(0, y) = 0 = v(\pi, y)$, $v(x, \pi) = x(\pi - x)$.

To this end find separation solutions $u(x, y) = X(x)Y(y)$ that satisfy the conditions $X(0) = 0 = X(\pi)$ and $Y(0) = 0$ (no condition for $Y(\pi)$). Call the various solutions thus found u_n where n is an integer index. Try to find an infinite linear combination $v(x, y) = \sum a_n u_n(x, y)$ that satisfies the nonhomogeneous BC $v(x, \pi) = x(\pi - x)$.

21. In the annulus (=ring) $r_0 \le r \le r_1$ we want to solve the PDE $\Delta u = 0$ with the BCs $u(r_0, \varphi) = 0$ and $u(r_1, \varphi) = U_0$. Find this solution u, then evaluate $\int |\nabla u|^2 \, r \, dr \, d\varphi$ for it. *In electrostatics, this is a capacitor problem: The outer plate $r = r_1$ is at voltage U_0, whereas the inner plate $r = r_0$ is at voltage 0. Such an arrangement of disconnected plates is called a capacitor. The integral $\int |\nabla|^2 \, r \, dr \, d\varphi$ denotes the energy stored in the electric field between the plates. This energy is $\frac{1}{2} C U_0^2$, where C is the called the capacity of the arrangement of plates. – In your physics course, you may have defined* *the capacity as ratio of charge and voltage. The present definition via the energy is equivalent. And yes, depending on the system of units you are using, there may be a few constants of nature in your definitions that are omitted in the discussion here.*

22. Solve $\Delta v = 0$ in the square $]0,\pi[\times]0,\pi[$ subject to the boundary conditions $v(x,0) = x$, $v(\pi,y) = \pi - y$, $v(x,\pi) = \pi - x$, $v(0,y) = y$. Hint: The method in Pblm. 20 displays that the separation works by putting *homogeneous* boundary conditions $u = 0$ on three sides of the square, using them already in choosing the separation solutions. So we write v as a sum of four solutions $v = v_t + v_b + v_l + v_r$, each of which satisfies *homogeneous* BCs $v = 0$ on three sides and and the BC prescribed in the problem for v on the fourth side. (The indices t, b, l, r stand for top, bottom, left, right.)

Note that each of the solutions v_t, v_b, v_l, v_r will be forced to have a discontinuity in one corner of the square, contrary to the pattern of the general theory that shuns such discontinuities on the boundary; but don't worry about the issue. It is not a troublespot in the specific example at hand.

23. The eigenvalue equation for the Laplacian is $-\Delta u = \lambda u$ in a domain Ω, with, say, $u = 0$ on $\partial\Omega$. The task is to find those numbers λ for which solutions other than the constant $u \equiv 0$ exist. Note that for a bounded domain Ω, such λ must be positive, because of the maximum principle. (Write down the details of this argument.) The λ for which nontrivial solutions exist are called eigenvalues, and the corresponding solutions u are called eigenfunctions.

It turns out that for a rectangle $\Omega = [0,a] \times [0,b]$, eigenvalues and corresponding eigenfunctions can be found by a separation of variables ansatz. Find them.

It turns out by some a closer study of Fourier theory and completeness arguments (which are not part or this problem here) that the separation method actually retrieves ALL *eigenvalues in this case.*

24. Write the expression

$$G(\vec{x}',\vec{x}) := \frac{1}{2\pi}\left(\ln|\vec{x}' - \vec{x}| - \ln\left|\frac{a}{|\vec{x}'|}\vec{x}' - \frac{|\vec{x}'|}{a}\vec{x}\right|\right)$$

in polar coordinates (r,φ), (r',φ') and show that its normal derivative on the boundary, namely $\frac{\partial}{\partial r'}|_{r'=a}G$, is indeed the Poisson kernel, which, for $a = 1$ at least, was studied in the lecture.

25. Find a_n such that $\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ for $0 \le x \le \pi$. – Also find b_n such that $\sin x = \sum_{n=1}^{\infty} b_n \sin 2nx$ for $-\pi/2 < x < \pi/2$. *These are not calculationally difficult problems, but they are designed to test whether you are already confusion-proof. If you have read the problem just superficially, read it again, very carefully.*

26. Show that the function $h(\vec{x}') := \left|\frac{a}{|\vec{x}'|}\vec{x}' - \frac{|\vec{x}'|}{a}\vec{x}\right|^{-1}$ is harmonic in the ball $|\vec{x}'| \le a$ in 3-space $\mathbb{R}^3$, for each $\vec{x}$ satisfying $|\vec{x}| < a$.

Write down the Green's function for the ball of radius a in $\mathbb{R}^3$ and take its normal derivative to obtain a Poisson integral formula for balls in $\mathbb{R}^3$.

27. (a) A rod of length π has initially temperature 100 and is then put in contact with ice (temperature 0) at both ends. But along its length, it is isolated. With the appropriate material constants, assume the heat equation is $u_t = \frac{1}{5}u_{xx}$. What is the temperature in the midpoint of the rod after time 2, numerically exact to 1 digit behind the decimal point? (You need to make a judgment how many terms in the infinite sum are needed for the required precision.)

(b) Now we allow the rod to be non-insulated along its length. So it will also loos heat according to Newton's law of cooling $u_t = -ku$, where we assume $k = \frac{1}{2}$. Both effects together are now described by the PDE $u_t = \frac{1}{5}u_{xx} - \frac{1}{2}u$. Solve this PDE by separation of variables, then answer the same question as in (a) for the new PDE.

28. A raw turkey goes from the fridge (40F) into a preheated oven (400F). So its temperature is 360F below the environment ($u = -360$). We want to know how long it takes until some core part of the turkey reaches 160F ($u = -240$, 240F below environment). Suppose for a small turkey, it takes 90 minutes; how long does it take for a large turkey? (A large turkey is 1.5 times as large, in each length, as a small turkey; so it is $1.5^3 = 3.375$ times as much food; it has the same shape as a small turkey).

So we discuss the same PDE $u_t = D\Delta u$ with BC $u = 0$ on $\partial\Omega$ on two different domains. Separation of variables is not appropriate for turkey shaped domains. But suppose $u(t,x)$ solves the PDE for the small turkey, which function v solves the same PDE for the large turkey? The answer shoud be $v(t,x) = u(?t, ?x)$, where the question marks are to be replaced by appropriate (possibly different) numbers. These numbers alone should suffice to give the answer, without solving the PDE explicitly.

Now we replace the small and large turkeys with a small and large freezer, the oven with a room, and the freezer warms up slowly due to a power outage. The large freezer is $\frac{3}{2}$ times as large in each dimension as the small freezer. But the freezers are insulated (good, but not perfect insulation): On the boundary of the small freezer, we therefore have the condition $\varepsilon u + \partial_\nu u = 0$, where ε is a small positive number (the smaller the better the insulation). On the large freezer, the boundary condition is either the same $\varepsilon u + \partial_\nu u = 0$ (same insulation as small freezer) or $\frac{2}{3}\varepsilon u + \partial_\nu u = 0$ (insulation $\frac{3}{2}$ times as thick as for small freezer). For the small freezer, we can tolerate 4 hours of power outage, until the inside temperature becomes intolerable for food preservation. How much time without power can we tolerate for the large freezer? Of the two BC's offered, choose the one that makes your life easier (which one?).

Note: Simple scaling arguments are the reason why smaller scale electric circuits pose less of a cooling problem than larger ones; why elephant sized ants in horror-SciFi movies would be biologically not feasible, why small birds eat their body weight in insects during a period of time in which folks like us could by no means do likewise, etc. Scaling arguments are not limitied to the heat equation. They explain roughly why small animals can jump from a height a multiple of their size, but large ones cannot. Wind channel experiments would be useless for the true specimen (airplane or whatever), if proper scaling were not taken into account.

29. The heat equation in a 3-dimensional ball is easier to solve than in a 2-dimensional disk. Assume radial symmetry: $u_t = \Delta u$ with $u = u(t,r)$. Separate variables (in the disk, as well as in the ball case). The ODE obtained in the ball can be simplified significantly by the substitution $R(r) = W(r)/r$. Note that at $r = 0$, you want the boundary condition $R'(0) = 0$ (why?). Solve the heat equation in a ball of radius r_0, with initial condition $u_0(r)$. Check the eigenfunction properties $\int_{\text{ball}} R_n(r) R_m(r) d\text{vol} = 0$ for $n \neq m$, remembering that $d\text{vol} = 4\pi r^2\, dr$ in the radially symmetric case.

Specifically, take a ball with diameter π, initial temperature 100, and $u_t = \frac{1}{5}\Delta u$. Fix the temperature on the boundary of the ball to 0. What is the temperature after time 2 in the center of the ball, to one digit after the decimal point?