UTK – M351 – Algebra I
Spring 2007, Jochen Denzler, MWF 10:10–11:00, Ayres 205

A Zoopark of homomorphisms

Examples & Hwk 31: If $\gcd(m,n)=1$, then the mapping $\mathbb{Z}_m \oplus \mathbb{Z}_n \to \mathbb{Z}_{mn}$ defined by $(\bar{a}_m, \bar{b}_n) \mapsto \bar{x}_{mn}$, according to the chinese remainder theorem, namely $x \equiv a \mod m$ and $x \equiv b \mod n$, is a ring isomorphism. Proofs in class.

In the following examples, write out what the claims about homomorphisms mean in formulas, prove them (some are trivial), or in the case where some mapping is not *a homomorphism / isomorphism etc., give a counterexample to the property in question. Wherever applicable, decide whether a given homomorphism is one-to-one, and/or onto.*

Examples & Hwk 32: If R and S are rings, then the mappings $i_1 : R \to R \oplus S$, $a \mapsto (a,0)$ and $i_2 : S \to R \oplus S$, $b \mapsto (0,b)$ are ring homomorphisms. Moreover, the mapping $i_D : R \to R \oplus R$, $a \mapsto (a,a)$ is a ring homomorphism.

The mapping $R \to R[i]$, $x \mapsto (x,0)$ is a ring homomorphism.

Example & Hwk 33: Given a ring R, is the mapping $x \mapsto -x$, $R \to R$ a ring automorphism?

Example & Hwk 34: The mapping $\mathbb{Z} \to \mathbb{Z}_n$, $x \mapsto \bar{x}_n$ (the congruence class of x modulo n) is a ring homomorphism.

Examples & Hwk 35: The mapping $\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$, $A \mapsto \det A$ is a group homomorphism. (Note that $\mathbb{R}^*$ is the group of all invertible real numbers (i.e., all real numbers other than 0), together with multiplication.) Is the mapping $\det : M_n(\mathbb{R}) \to \mathbb{R}$, $A \mapsto \det A$ a ring homomorphism?

Example & Hwk 36: Given any interval $[a,b] \subset \mathbb{R}$ (with $b > a$), the mapping $I : C^0[0,1] \to C^0[a,b]$ given by $(I(f))(x) := f(\frac{x-a}{b-a})$ for $a \le x \le b$ is a ring isomorphism.

If $[0,1] \supseteq [a,b]$, then the mapping $J : f \mapsto f|_{[a,b]}$ which restricts the domain of a function f to the smaller domain $[a,b]$ is a ring homomorphism. It is onto, but (unless $[a,b] = [0,1]$) not 1-to-1.

Example & Hwk 37: Given any ring R and any set X, the set of all functions $f : X \to R$, together with addition and multiplication defined by $(f+g)(x) := f(x)+g(x)$, $(fg)(x) := f(x)g(x)$, is a ring. (Can you prove this?) We sometimes call this ring R^X.

The mapping $\chi : \mathcal{P}(M) \to (\mathbb{Z}_2)^M$, $A \mapsto \chi_A$, where the function χ_A is defined by $\chi_A(x) = 0$ if $x \notin A$ and $\chi_A(x) = 1$ if $x \in A$, is a ring isomorphism. (The main difficulty in proving this statement is to carefully understand what is being said. And the best test of whether you have understood the statement is whether you can reproduce it closed-notes, without rote memorization.)

Example & Hwk 38: The map $R \to M_2(R)$, $x \mapsto \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ is a ring homomorphism.

Example & Hwk 39: The map $\mathbb{C}^* \to GL_2(\mathbb{R})$, $\alpha + i\beta \mapsto \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ is a group homomorphism.

Example & Hwk 40: Take the group consisting of the six rational functions $I, F_0, F_1, F_\infty, M, W$ discussed in Ex&Hwk. 8, and the group S_3 discussed in Ex.&Hwk. 7. Write down the group tables and show that these two groups are isomorphic, by explicitly exhibiting a group isomorphism.

Example & Hwk 41: Continuing with the group RF of all rational functions f of the form $f(z) = (az+b)/(cz+d)$, with composition, from Ex&Hwk 8, take the mapping $P : GL_2(\mathbb{R}) \to RF$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto f_{abcd}$, where $f_{abcd}(z) := (az+b)/(cz+d)$. This map is a group homomorphism. Is it 1-to-1? In order to answer whether it is onto, you may need to make the statement more precise: do we mean *real* or *complex* rational functions?