UTK – M351 – Algebra I
Spring 2007, Jochen Denzler, MWF 10:10–11:00, Ayres 205

A Zoopark of groups

Examples & Hwk 1: $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{R},+)$ and $(\mathbb{C},+)$ are groups. — $(\mathbb{N},+)$ is NOT a group. *Why not? — Is the set of irrational numbers, with +, a group? Explain.*

Examples & Hwk 2: $(\mathbb{Z}^*,\cdot)$ is NOT a group. *Why not?* — $(\mathbb{Q}^*,\cdot)$, $(\mathbb{R}^*,\cdot)$ and $(\mathbb{C}^*,\cdot)$ are groups. Here we have used the following notations for certain sets of numbers: $\mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$, $\mathbb{Q}^* := \mathbb{Q}\setminus\{0\}$, $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$, $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$.

Examples & Hwk 3: Matrix groups. We denote by $GL_n(\mathbb{R})$ the set of all invertible $n\times n$ matrices with real entries, and by $SL_n(\mathbb{R})$ the set of all those $n\times n$ matrices with real entries whose determinant is 1. By $O_n(\mathbb{R})$ we denote the set of orthogonal $n\times n$ matrices with real entries. Similarly, we write $GL_n(\mathbb{Q})$ or $GL_n(\mathbb{Z})$ etc. to require that the entries be rational numbers, or integers. $+$ and $\cdot$ denote matrix addition or multiplication respectively. — $(GL_n(\mathbb{R}),\cdot)$ is a group. *Why? Which statement from Math 251 is pertinent here? Are $SL_n(\mathbb{R})$ and $O_n(\mathbb{R})$ groups? How about $GL_n(\mathbb{Q})$, $GL_n(\mathbb{Z})$, $SL_n(\mathbb{Q})$, $SL_n(\mathbb{Z})$, each together with $\cdot$?*

Example & Hwk 4: The set $\{E,O\}$ together with addition subject to the rules $E+E=E$, $E+O=O+E=O$, $O+O=E$ is a group. *Can you recognize some simple wisdom commonly known to high school graduates that is represented by this abstract example?*

Example & Hwk 5: Can you generalize the previous example and construct a group on a set of three elements, let's call it $\{Z,I,T\}$, with an appropriate addition, with a similar idea behind the abstract example?

Example & Hwk 6: The set $\{a+b\sqrt{2}\mid a,b\in\mathbb{Q}\}\setminus\{0\}$, together with multiplication, is a group. *Show this. Explain why it is necessary to know that $\sqrt{2}$ is irrational in order for your proof to work.*

Example & Hwk 7: The set of permutations of $\{1,2,\ldots,n\}$, together with composition $\circ$, is a group, often called S_n. Remember the definition of permutations: For instance, the permutation (1347256) is the function $1\mapsto 1, 2\mapsto 3, 3\mapsto 4, 4\mapsto 7, 5\mapsto 2, 6\mapsto 5, 7\mapsto 6$. What is the inverse of this permutation? What is $(1347256)\circ(3512647)$? What is $(3512647)\circ(1347256)$?

Example & Hwk 8: Take the following set of six rational functions $\{I,F_0,F_1,F_\infty,M,W\}$ given by $I(z)=z$, $F_0(z)=z/(z-1)$, $F_1(z)=1/z$, $F_\infty(z)=1-z$, $M(z)=(z-1)/z$, $W(z)=1/(1-z)$. Don't try to understand the rationale behind the names. Use composition $\circ$ as operation. This is a group. *To help showing this, construct the group table.*

Is the set of all rational functions, together with $\circ$, a group? Give a reason for your answer. — Show that the set of those rational functions of the form $f(z)=\frac{az+b}{cz+d}$ with a,b,c,d real and $ad-bc\neq 0$ forms a group, again under the composition $\circ$ as operation.

Example & Hwk 9: Draw a regular n-gon, say, to be specific, a hexagon $n=6$. Draw its six symmetry axes $a_1,\ldots,a_6$ (half of them are lines through opposite vertices, half of them are lines through midpoints of opposite sides. Let $S_1,\ldots,S_6$ be reflections of the hexagon with respect to the axes $a_1,\ldots,a_6$ respectively. Moreover, let $R_0,\ldots,R_5$ denote counterclockwise rotations about the center of the hexagon by 0, 60, 120,…,300 degrees resectively. The mappings $S_1,\ldots,S_6,R_0,\ldots,R_5$, together with composition, form a group, which is called DIHEDRAL GROUP D_6. (Or similarly D_n, if you have an n-gon instead of a hexagon.)

Example & Hwk 10: The set $O_2(\mathbb{R})\times\mathbb{R}^2$ consists of pairs (A,a) where A is an orthogonal 2×2 matrix and a is a 2-vector. We define the multiplication by $(A,a)\odot(B,b)=(AB,Ab+a)$. Check that this is a group. It is called the symmetry group of the plane, because the motiviation behind the definition of $\odot$ is that (A,a) represents the mapping $x\mapsto Ax+a, \mathbb{R}^2\to\mathbb{R}^2$.