Hints for Problem 24

Here is an outline of what you should do:

We want to find $Q(Q(Q(4444^{4444})))$. The naive way (which turns out utterly unfeasible) would be: calculate 4444^{4444}, add up all its digits, write down the result, which is $Q(4444^{4444})$, add up all the digits of this result, obtain $Q(Q(4444^{4444}))$, and then do it again.

This is utterly unfeasible since 4444^{4444} is so huge that it doesn't fit onto your pocket calculator. *Get a rough idea how many digits* 4444^{4444} *has.* (Fancy how many pages it would need to write them all down with 80 digits per line and 60 lines per page!)

You want an *upper bound* for the number of digits of 4444^{4444}: "at most that many". And you want to conclude: since 4444^{4444} has at most ***** digits, $Q(4444^{4444})$ (which is the sum of its digits) can be at most *****. Since it is at most *****, it has at most **** digits and therefore the $Q(\cdot)$ of this number can be at most ****. Continue that way until you get a result of the form $Q(Q(Q(4444^{4444}))) \leq$ *some small number*. **That's part 1 of your job**

Next you should remember that for any positive integer x, it holds $Q(x) \equiv x \pmod 9$. Maybe I need to remind some of you that this is what we discussed in class. For instance, for the number 368294585127, we have $Q(368294585127) = 3+6+8+2+9+4+5+8+5+1+2+7 = 60$, and $368294585127 \equiv 60 \pmod 9$. This result holds in particular for $x = 4444^{4444}$, whence $Q(4444^{4444}) \equiv 4444^{4444} \pmod 9$. It also holds for $x = Q(4444^{4444})$, whence $Q(Q(4444^{4444})) \equiv Q(4444^{4444}) \pmod 9$. And you do the next step.

This means: If you can find in which congruence class ([0], [1], [2], [3], [4], [5], [6], [7], [8]) mod 9 the number 4444^{4444} is, you know that $Q(Q(Q(4444^{4444})))$ is in the same congrunece class.

The punchline now is: Even though 4444^{4444} is prohibitively huge, you can still figure out its congruence class mod 9 quite easily. To this end I may need to remind you that we showed in class: If $a_1 \equiv a_2 \pmod 9$ and $b_1 \equiv b_2 \pmod 9$, then $a_1 b_1 \equiv a_2 b_2 \pmod 9$. This means in particular:

Since $4444 \equiv ? \pmod 9$ (*you figure out the question mark*), we have $4444 \cdot 4444 \equiv ?\cdot? \pmod 9$, and continuing $4444 \cdot 4444 \cdot 4444 \equiv ?\cdot?\cdot? \pmod 9$ etc. Inductively, if $a_1 \equiv a_2 \pmod 9$, then $a_1^n \equiv a_2^n \pmod 9$. for all natural numbers n.

Now go ahead finding the congruence class (mod 9) of 4444^{4444}; in your step by step calculation, you may replace numbers that become large by smaller numbers in the same congruence class. **That gives the second piece of information.**

Take the two pieces of information together and answer the question from the problem.