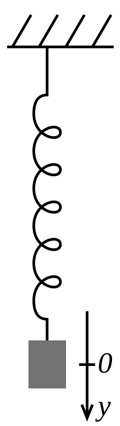
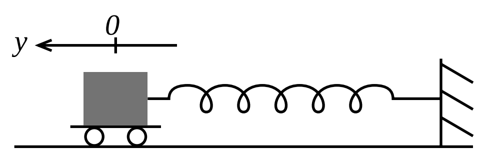
UTK – M231 – Differential Equations – Jochen Denzler
Applications of Chapter 4: Forced oscillations and resonance

We are going to use the theory of 2nd order constant coefficient linear ODEs to study oscillation phenomena. This is is the most important application of Chapter 4, maybe of the entire course.

Think of a mass suspended from a spring. The fact that this simple construction is not seen realized in everyday life shouldn't distract you. With more or less precision, virtually all phenomena where some object vibrates or oscillates can be described by the same (or very similar) differential equations: vibrating bridges or other concrete structures, the perpendicle in a grandfather's clock. Moreover, electronic circuits constructed from a capacitor and an inductivity (and the ever-present Ohm resistance), and used to create and amplify electromagnetic waves like the ones carrying radio signals, are also goverened by the same underlying equations. Filtering high frequency or low frequency signals out of a signal containing a mixture of frequencies is based on the fact that different frequencies are amplified more or less, depending on the parameters of the oscillation mechanism. You are advised to reread this introductory paragraph once you are through with the technical discussion to follow now.



We are considering a mass m suspended from a spring. We choose our coordinate y in such a way that $y = 0$ corresponds NOT to the unexpanded spring, but to the spring expanded just as much such as to balance the weight. Any excess expansion $y > 0$ will then correspond to a pulling force of the spring exceeding the weight of the object and thus pulling it back up. By referring to the force ky from the spring, according to Hooke's law, we are only considering the excess force beyond the one that balances gravity, because our y is also only the excess expansion of the spring beyond what is needed to balance gravity. This is why in our equations, gravity does not show up. It is tucked away neatly in a smart choice of coordinates; and this is just fine, because then it doesn't get in our way when we focus our study on the oscillations. (If you do not like this idea of hiding gravity, and it looks more like black magic to you, you may instead consider the horizontal version as depicted in the second figure.)



First consider the mechanism in the absence of friction (air resistance): The ODE governing it is $my'' + ky = 0$, according to Newton's law. The solutions are the **F**ree **U**ndamped **O**scillations

$$y = A\cos\omega_0 t + B\sin\omega_0 t \qquad \text{with} \qquad \omega_0^2 = \frac{k}{m} \tag{FUO}$$

The wisdom of exposition consists of expressing the solutions in terms of easily observable parameters here, and interpreting their meaning. So ω_0 is the frequency with which the oscillator oscillates when kicked and then left alone.

One of the important things to understand about trig functions in this context is the following: any linear combination of sine and cosine with the *same* frequency can be written as a sine alone, with an appropriate amplitude R, and an appropriate phase shift. So, given any numbers A, B, you can calculate numbers R, ϕ from them such that

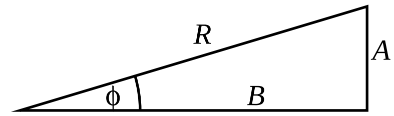
$$A\cos\omega_0 t + B\sin\omega_0 t = R\sin(\omega_0 t + \phi) \tag{1}$$

becomes a true identity. Indeed, using the addition theorem $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ on the right hand side, you see that

$$R\sin(\omega_0 t + \phi) = R\sin(\omega_0 t)\cos\phi + R\cos(\omega_0 t)\sin\phi$$

So, in order for (1) indeed to be true, you have to find R and ϕ in such a way that

$$R\sin\phi = A\,, \qquad R\cos\phi = B$$



This means you need $R = \sqrt{A^2+B^2}$ and $\tan\phi = A/B$.

Using this little calculation, we prefer to rewrite the solution (FUO) as

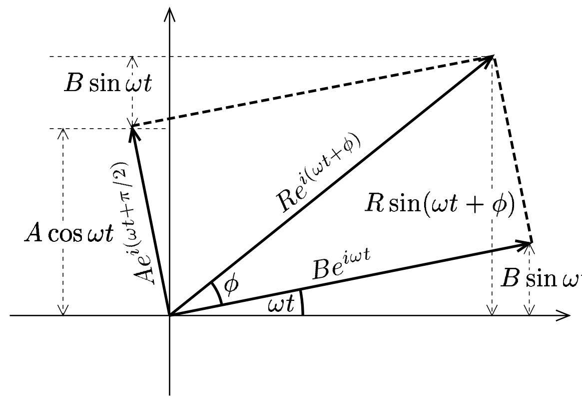
$$y(t) = R\sin(\omega_0 t + \phi) \qquad \text{with} \qquad \omega_0^2 = \frac{k}{m} \tag{FUO$'$}$$

because R (unlike A and B) is a very conspicuous quantity, namely the amplitude of the oscillation.

If you have done the homework, you know how the same formula (1) can be obtained using complex numbers; giving this second, alternative explanation for that fundamental formula is very instructive. *First look at the figure of the right triangle on the previous page.* But now pretend that this triangle lies in a plane representing complex numbers. The corner at which the angle ϕ is measured is the origin. The right angle of the triangle lies at the real number B, and the remaining corner of the triangle lies at the complex number $B + iA$. We see that $B + iA = R\cos\phi + iR\sin\phi = R(\cos\phi + i\sin\phi) = Re^{i\phi}$. It is therefore easy to locate a complex number given in the form $Re^{i\phi}$ in the complex number plane: from the origin, go a distance R in the direction specified by the angle ϕ.

Concerning formula (1), you can add $A\cos\omega t$ and $B\sin\omega t$ geometrically by first adding (as vectors in the plane of complex numbers) $Ae^{i(\omega t+\pi/2)}$ and $Be^{i\omega t}$, and then reading off their imaginary parts on the vertical axis (see new figure now):

$$\begin{aligned} \sin\omega t &= \operatorname{Im} e^{i\omega t} \\ \cos\omega t &= \operatorname{Im} e^{i(\omega t+\pi/2)} \\ A\cos\omega t + B\sin\omega t &= \operatorname{Im}\left(Be^{i\omega t} + Ae^{i(\omega t+\pi/2)}\right) \\ &= \operatorname{Im}\left(Re^{i(\omega t+\phi)}\right) \\ \text{with} \quad & R\sin\phi = A \\ & R\cos\phi = B \end{aligned}$$



The occurrence of the right angle makes it clear why the amplitudes A and B for the sine and cosine respectively are combined according to the law of Pythagoras into a total amplitude $R = \sqrt{A^2 + B^2}$.

Now that you have understood how to turn an expression of the form $A\cos\omega t + B\sin\omega t$ into an equivalent one of the form $R\sin(\omega t + \phi)$, let us resume our discussion of the oscillator. Our first refinement is to add a small friction (air resistance) to the model. The ODE then reads

$$my'' + by' + ky = 0$$

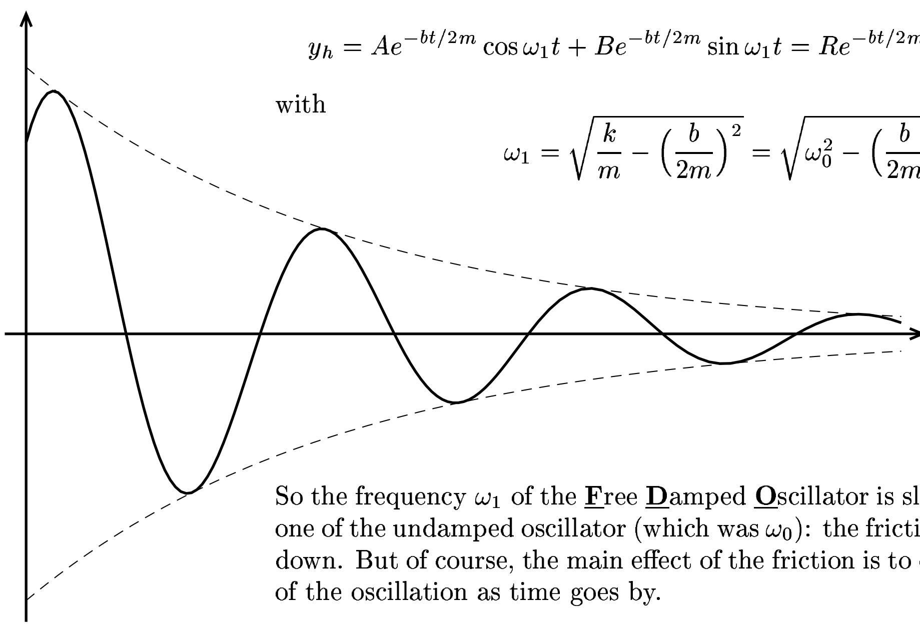
with b sufficiently small (namely $b < \sqrt{4mk}$, i.e., $\frac{b}{m} < 2\omega_0$); its general solution is

$$y_h = Ae^{-bt/2m}\cos\omega_1 t + Be^{-bt/2m}\sin\omega_1 t = Re^{-bt/2m}\sin(\omega_1 t + \phi) \qquad \text{(FDO)}$$

with

$$\omega_1 = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

So the frequency ω_1 of the **F**ree **D**amped **O**scillator is slightly smaller than the one of the undamped oscillator (which was ω_0): the friction slows the oscillation down. But of course, the main effect of the friction is to decrease the amplitude of the oscillation as time goes by.



Notice that we are only discussing the case of small damping here (namely $b < \sqrt{4mk}$). It is characterized by complex conjugate roots of the auxiliary equation and actually shows oscillatory solutions. The case of large damping (namely $b > \sqrt{4mk}$) yields real roots in the auxiliary equation, and the solutions will no longer oscillate. The shocks in your car are (hopefully) designed that way. This large damping case (as well as the borderline case $b = \sqrt{4mk}$) is discussed in the book, but I leave it out here. If you really understand the weakly damped case discussed in these notes, you can study the strongly damped case on your own, even without the book; whereas, if you don't, there's no point anyway in smiting you with a second example, and I'd rather suggest you focus on the weakly damped case.

We now discuss the forced oscillator, i.e., we apply some external force to the system; and here, we are going to assume that this force is of the form $F_0 \sin \gamma t$ with a frequency γ. We will mainly study how the system reacts to forcing with different frequencies γ. In electronic circuits, $F_0 \sin \gamma t$ could be an input voltage (AC), and γ would likely be $2\pi \times 60/\mathit{second}$. There are other important cases, in which the forcing is periodic, but not given by a trig function, like the zigzag function whose graph looks like this: . However, it was claimed by the French mathematician Fourier in 1807, that "every" periodic function with frequency γ can be written as a linear combination of (infinitely) many sine and cosine functions with different frequencies: namely γ, 2γ, 3γ,.... This surprising claim caused much dispute at the time, but has been found true in the decades following (subject to a bunch of footnotes making that claim technically precise). With Fourier's observation in mind, and using the superposition principle, a discussion only of forcing terms of the form $F_0 \sin \gamma t$ is actually of a much broader use than one would anticipate at first sight.

For the forced oscillator, we have the ODE

$$my'' + by' + ky = F_0 \sin \gamma t = \frac{F_0}{2i}(e^{i\gamma t} - e^{-i\gamma t})$$

and we continue to assume small damping $0 < b < \sqrt{4mk}$. We need to look for a particular solution y_p of the inhomogenuous equation. The general solution is then $y_p + y_h = y_p + Re^{-bt/2m} \sin(\omega_1 t + \phi)$. The second term goes to 0 as $t \to \infty$, i.e., it will become negligibly small after a long period of time has elapsed. In the following, we do not focus on initial behavior of the system, but on the long term behavior (i.e., after the contribution from y_h has faded away). This is given precisely by the particular solution y_p alone, and we now study how to find it:

The particular solution we are finding by the method of undetermined coefficients here is exactly the one that describes the long range behavior after all initial effects have faded away.

As we have a somewhat messy calculation awaiting us (mainly due to the fact that we have to carry through the parameters m, b, k rather than replacing them by numbers especially preselected for convenience), it is convenient to use the complex method of undetermined coefficients and to try $y = A_+ e^{i\gamma t} + A_- e^{-i\gamma t}$. Plugging this into our equation, we find

$$A_+\Big(m(i\gamma)^2 + b(i\gamma) + k\Big) = \frac{F_0}{2i} \qquad \text{hence } A_+ = \frac{-\frac{F_0}{m}}{2\Big(\frac{b}{m}\gamma - i(\frac{k}{m} - \gamma^2)\Big)}$$

$$A_-\Big(m(-i\gamma)^2 + b(-i\gamma) + k\Big) = \frac{-F_0}{2i} \quad \text{hence } A_- = \frac{-\frac{F_0}{m}}{2\Big(\frac{b}{m}\gamma + i(\frac{k}{m} - \gamma^2)\Big)}$$

You may find that I have written the result for A_+ and A_- in somewhat too messy a way. If you do the calculation for yourself, you will see that I have deliberately divided numerator and denominator by m, which turned a simple fraction into a fraction of fractions. But I have a good reason for doing this: I prefer that the parameters b and k do not stand alone, but rather are always combined with m in the form b/m and k/m. This resumes the wisdom of exposition mentioned two pages ago, namely that the equations should highlight *practically conspicuous* quantities. Remember that $k/m = \omega_0^2$, the square of the frequency of the free undamped oscillator, and that b/m determines how fast the damped oscillations decay; indeed it is the combination b/m that occurs in the exponential $e^{-(b/m)(t/2)}$ in the solution of the free damped oscillator (FDO). So it is the quantities b/m and k/m that tell us what the oscillator would be capable of doing if left alone. The behavior of the *forced* oscillator will depend a lot on whether our forcing frequency γ is close to the oscillator's "own" frequency ω_0 or not.

The denominators of A_+ and A_- can be written in polar coordinates, as we have done before:

$$\frac{b}{m}\gamma + i(\frac{k}{m} - \gamma^2) = re^{i\theta} \quad \text{with} \quad r = \left(\Big(\frac{b}{m}\gamma\Big)^2 + \Big(\frac{k}{m} - \gamma^2\Big)^2\right)^{1/2} \quad , \quad \theta = \arctan \frac{\frac{k}{m} - \gamma^2}{\frac{b}{m}\gamma}$$

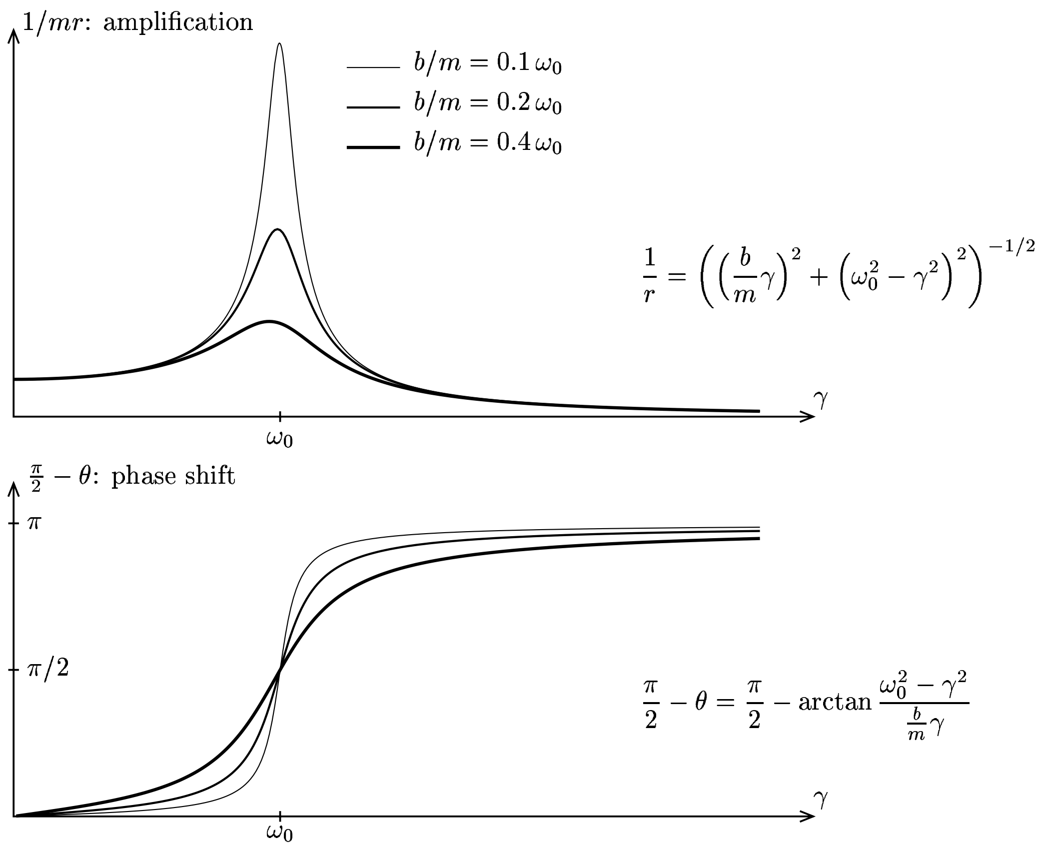
This trick speeds up the remaining evaluation of y_p significantly:

$$y_p = A_+ e^{i\gamma t} + A_- e^{-i\gamma t} = \frac{-F_0}{2mr}(e^{i(\gamma t + \theta)} + e^{-i(\gamma t + \theta)}) = \frac{-F_0}{mr}\cos(\gamma t + \theta) = \underline{\underline{\frac{F_0}{mr}\sin\Big(\gamma t - (\frac{\pi}{2} - \theta)\Big)}}$$

Now have a look at this result from an input-output point of view: The forcing (inhomogeneity) is our *input* into the oscillator (the spring mechanism, or an electronic circuit). The solution of the ODE is how the oscillator responds to this forcing, namely its output:

$$\underset{\text{Input}}{\boxed{F_0 \sin \gamma t}} \longrightarrow \underset{\text{Output}}{\boxed{\tfrac{F_0}{mr} \sin\Big(\gamma t - (\tfrac{\pi}{2} - \theta)\Big)}}$$

The input amplitude F_0 transforms into an output amplitude $F_0/(mr)$, so you have an amplification factor $1/(mr)$, which we'll discuss in a moment. Moreover, there is a phase shift $\frac{\pi}{2} - \theta$, which may well be non-zero. If your input forcing tries to push the oscillating mass up, that doesn't mean the mass will go up immediately. It may very well lag behind. How much it lags behind will depend on the relation between the forcing frequency γ and the oscillator's own frequency ω_0, as we will now see: We recapitulate the formulas for the amplification factor and the phase shift from above, and we plot them, as functions of the forcing frequency.



The following things can be learned from the diagram and its formulas:

Resonance:

The amplification is particularly large, if the forcing frequency γ is close to the frequency ω_0 of the unforced oscillator. The precise frequency of maximal amplification turns out to be $\gamma_* = \sqrt{\omega_0^2 - 2(\frac{b}{2m})^2}$ (a brief calculus minimax problem). The amplication at this frequency is $(\frac{b}{m}\omega_1)^{-1}$, i.e., inversely proportional to the damping parameter $\frac{b}{m}$. Therefore this resonance effect is less pronounced if there is a large damping, but becomes very strong as the damping goes to 0. — For very high forcing frequency, the input is essentially absorbed by the oscillator, i.e., the amplification factor goes to 0.

Phase shift:

For very high forcing frequency, the oscillator shows a phase shift of π, i.e., it always moves in the opposite direction from the one in which you push. The phenomenon is a bit similar to the situation when the weather changes much more rapidly than your usual time of changing clothes: you may end up wearing a sweater today (based on your experience that you had been freezing yesterday), but it is already hot again today. So you decide to wear a T-shirt tomorrow, but tomorrow it will turn out to be darn cold again :-) In contrast, if the forcing frequency is very low, compared to the frequency of the unforced oscillator, then the oscillator will have no trouble following the forcing, and the phase shift is nearly 0. At resonance ($\gamma = \omega_0$) the phase shift is $\pi/2$.

Near resonance transition region:
You can view this transition region best, when looking at the phase shift. If you have small damping, the phase shift changes very swiftly near the resonance frequency. The oscillator has a very clear-cut and decisive distinction between what it considers as 'fast' forcing (phase shift almost π) and 'slow' forcing (phase shift almost 0). It is only a small interval around the resonance frequency where there is an intermediate phase shift. If you have a larger damping, this transition region becomes wider. The more strongly damped oscillator is much less picky about the precise forcing frequency than the oscillator with very small damping.

Stronger damping:
Whereas there is a clearcut distinction $b >/< \sqrt{4mk}$ for the general solution y_h of the homogeneous equation (namely, the free, or unforced, oscillator), the particular solution y_p of the forced oscillator does not require a similar distinction of three cases. The only change that may be perceived as qualitatively significant is at $b = \sqrt{2mk}$, i.e., $\frac{b}{m} = \sqrt{2}\omega_0$, at which parameter the amplification curve ceases to have a maximum γ_*.