UTK – M231 – Differential Equations – Jochen Denzler
Notes on Chapter 4: Second Order Linear ODEs

What we are going to learn on (2nd order) *linear* ODEs makes much more sense as a whole than in the piecemeal way in which you necessarily have to learn it. This is why I deviate from the book significantly, as far as the presentation is concerned, even though the contents is basically the same.

We are dealing now with **2nd order** ODEs, that is, ODEs containing second derivatives. So we look for y, as a function of t, from an equality involving y'' and any or all of y', y, t. In practice, we don't (and couldn't) handle such a vast generality. For instance $2y'' + \sin y'' = y^2 + t^2$ is a legitimate 2nd order ODE which we won't touch. Rather, we want cases where we could isolate the highest (i.e., second) derivative on one side of the equation. As, for instance, $y'' + \sin(y) = \cos(5t)$, which could be rewritten as $y'' = -\sin(y) + \cos(5t)$.

Actually, this is still too general and too complicated; we will deal exclusively with *linear* second order ODEs here.[1] Recall what we mean by a *linear* ODE (whatever its order):

An (O)DE is *linear*, if the unknown function (in other words: the dependent variable) occurs linearly in the equation, that is, assuming the dependent variable is y and the independent variable is t (the unknown function being $t \mapsto y(t)$), we have only $y, y', y'' \ldots$, possibly multiplied by given expressions of t, and added together.
So the eqn should NOT contain: y^2, y'^2, $\sin y$, $y \cdot y'$, $1/(1+y)$, ...
but ok are: t^2, $\sin t$, $e^t \cdot y$, $e^t y'/(1+t)$, y''', ...

Therefore, a linear 2nd order ODE can be written as

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

where $p(t)$, $q(t)$, $g(t)$ are any given expressions involving only the independent variable t, but NOT y nor its derivatives. Of course, some straightforward algebraic manipulations may be required first to bring the equation into this form.

We need another two pieces of language:
(I) We call a linear ODE **homogeneous**, if $g(t) \equiv 0$, i.e., if it fits into the paradigm $y'' + p(t)y' + q(t)y = 0$.

> *WARNING:* This is a hideous use of language, for which I decline responsibility, because I have not invented it. The word homogeneous is used in a completely different meaning here than in sec 2.6!!! — If g doesn't vanish, we call the eqn (1) linear *in*homogeneous, and the term $g(t)$ will be called the inhomogeneity. Make sure you understand these words, because almost the whole chapter hinges on them. Henceforth, homogeneous will refer to the new meaning, unless explicitly specified otherwise. If need arises to distinguish the two meanings of homogeneous, I'll say "linear homogeneous" for the new meaning and "$v = y/x$-homogeneous" for the old meaning.

(II) We say that ODE (1) has **constant coefficients**, if both $p(t)$ and $q(t)$ (namely the coefficients of y' and y respectively) are constant, i.e., just numbers, not actually depending on t. Otherwise we say the ODE has variable (or nonconstant) coefficients. (The inhomogeneity $g(t)$ is NOT anybody's coefficient and is therefore NOT required to be constant.)

So we will study 2nd order linear ODEs of type (1), and we will give a more thorough study to the special case of 2nd order linear ODEs with constant coefficients, but only briefly mention the case of non-constant coefficients.

Note that the above definitions generalize to higher order linear ODEs (we may occasionally mention them) and also to lower (i.e., first) order ODEs. The latter is relevant, because we are going to compare the new material with what we know about 1st order linear ODEs, and this backwards comparison is alas almost absent in the book.

Examples:

$y'' + y' - 11ty = t^2$	2nd order, linear inhomogeneous, variable coeff's
$y'' + 5y' - 7y = \sin 2t$	2nd order, linear inhomogeneous, constant coeff's
$y'' + t^3 y' - 7y = 0$	2nd order, linear homogeneous, variable coeff's
$2y' - 5y - 3t^2 = 0$	1st order, linear inhomogeneous, constant coeff's
$y' - t^2 y = 0$	1st order, linear homogeneous, variable coeff's
$y''' - 22y'' + 5y' - 7y = e^t$	3rd order, linear inhomogeneous, constant coeff's
$y^{(4)} - ty''' + 2y' - 9y = 0$	4th order, linear homogeneous, variable coeff's

[1] The only nonlinear 2nd order ODEs in this class is of a type discussed earlier; namely $y'' = f(y)$ coming from Newton's law of motion. The trick was to use y' as an integrating factor; see my notes on Ch. 2, after 2.3.

I urge you to refer back to this and the following two pages, possibly rereading it, whenever we have covered a section of chapter 4 in the book, or whenever a homework trains one of the methods announced here. This should make sure that all the details you are learning will stand in a larger context right away.

The basic principle about linear ODEs (actually about linear equations in general, never mind if they are ODEs, PDEs, or systems of linear (algebraic) equations) is the **superposition principle**, which you will learn soon. One of its variants says the following: The task to find **all** solutions of a linear **in**homogeneous equation can be split into two smaller tasks: (a) find **all** solutions to the corresponding **homogeneous** equation, and (b) find **some** solution (maybe by guessing) to the **in**homogeneous equation.

If you think words like 'all' or 'some' aren't worth bothering about, make a U-turn immediately! The distinction between 'all' and 'some' is critical here, as it is in all of mathematics (even though they fail to tell you so in some math courses)

Now look at the section headings in the table of contents in the book:

First some ODE lingo: When they say 'the general solution' they do NOT really mean a single solution (ranking much higher than a colonel or even a sergeant solution;–), but they really mean **all** solutions; but since they give all of them in one single formula, they call it the general solution. So the general solution of $y'(t) = 2y(t)$ is $y(t) = C\,e^{2t}$ where any choice of a real number for C gives a solution.

Next a warning: Section headings in obese textbooks reflect serving sizes, even at the expense of doing violence to the logical structure. This is a *general* warning, because it applies to (almost) *all* obese textbooks. So let's sort this out, because I'll teach this chapter by logical structure, not by serving sizes.

LOGICAL STRUCTURE	TEXTBOOK CHAPTER [and comments]
• The key example serves as an introduction	Introduction: The Mass–Spring–Oscillator
• How to find all solutions to a linear homogeneous ODE with constant coefficients:	
– The superposition principle and the auxiliary equation – auxiliary equations with real roots	Homogeneous Linear Equations: The General Solution
– auxiliary equations with complex roots	Auxiliary Equations with Complex Roots
• How to find some solutions to a linear inhomogeneous ODE:	
– Undetermined coefficients (works for const coeff ODEs only)	Nonhomogeneeous Equations: The Method of Undetermined Coefficients
– Variation of parameters (works rather generally)	Variation of Parameters [This is covered a bit later]
• How to put the preceding two bullets together to get all solutions of an inhomogeneous ODE	The Superposition Principle and Undetermined Coefficients Revisited [But it's really all about the superposition principle]
• The key example in detail: Mechanical (or electrical) oscillations	[you may say vibration in mechanics — oscillation is the same thing in more general context]
– Free oscillations (homogeneous ODE)	A Closer Look at Free Mechanical Vibrations
– Forced oscillations (inhomog' ODE)	A Closer Look at Forced Mech' Vibrations

On the next page, you'll find the same logical outline in table form. When you see how much work I have put in these tables, you can guess how important I think it is that you understand how everything fits together.

This entire logical structure, namely the superposition principle (apart from the 'how to' recipes) is characteristic of *linear* equations, be they ODEs or PDEs or systems of linear equations like you study in a Linear Algebra / Matrix Algebra course.

Those of you who have taken M251 already, should discover similarities with material studied there. Those who take M251 concurrently should watch out for these similarities. Those who will take M251 later should make sure to refer back to these present notes to see the similarity at some future time. I have posted some material form M251 on the web that stresses the analogy from the M251 point of view.

So here we go again: How is the the task of finding all solutions to a linear inhomogeneous ODE split in two separate, largely independent tasks?

Find **all** solutions to a linear **in**homogeneous ODE,
e.g., 2nd order: $y'' + p(t)y' + q(t)y = g(t)$

↙ ↘

Find **all** solutions to the corresponding linear **homogeneous** ODE,
e.g., 2nd order: $y'' + p(t)y' + q(t)y = \mathbf{0}$

How will we do it?

- If we have constant coefficients, a routine method is available, which is discussed in Sec 4.2–3. The method says, roughly, and still subject to some embellishments: "Solutions to this type of equations will be $y = e^{rt}$, you only have to find r by plugging in." You have already encountered a sample of this method, out of context: it was hwk 1.2#20. This is an instance of "educated guessing": The education tells you to guess some kind of exponential, but you still need to figure out r.

- For very particular types of variable coefficients, a similar routine method is available. We don't discuss this type (so-called Euler equations) in detail here. But you have already encountered a sample of this case as well, out of context: it was hwk 1.2#21.

- For the general case of variable coefficients, the task may be difficult, and we just don't have a general method. However, power series methods (see Chapter 8) are very helpful in the practically important cases, and if you ever run across "Bessel functions" (quite likely) or "hypergeometric functions" (less likely), this is the context where they arise. [Not covered in this class.]

Find **some** solution to the original linear **in**homogeneous ODE,
e.g.,2nd order: $y'' + p(t)y' + q(t)y = g(t)$

How will we do it?

- If all you need is just **some** solution, guessing or eyeballing may be a useful method already.

- A jazzed-up version of guessing is *educated* guessing: For constant coefficients, with only sine, cosine, exponentials and powers as inhomogeneity, it works fine. Section 4.4 does it, under the name **Method of undetermined coefficients**. Here, "undetermined coefficients" refers to coefficients in the (guessed) solution, whereas "constant coefficients" refers to coefficients in the equation.

- For other inhomogeneities (even with constant coefficients), and also for variable coefficients (even with nice inhomogeneities), educated guessing rarely works. The method of choice is called **variation of parameters** (Sec. 4.6). I won't summarize its contents here. Be it known however that it requires to solve the corresponding homogeneous equation first.

  You could of course use this method also for the simpler cases mentioned before; but "undetermined coefficients" will be faster, where it works.

I stressed above that "2nd order" is not the key issue here, but "linear" is. Therefore, all of the above also applies to 1st order linear equations, and therefore you should be able to recognize some known stuff, if you try to treat first order linear equations by the methods outlined above. Probably, as you begin, you will not find much that looks familiar. But let's have a closer look:

A first order linear **in**homogeneous ODE looks as follows:
$y' + p(t)y = g(t)$

↙ ↘

When you try to find an integrating factor, you indeed don't care about the inhomogeneity $g(t)$ yet. But do you really solve the **homogeneous** ODE, i.e., $y' + p(t)y = \mathbf{0}$?

The answer is yes, and this is your homework N4#1:
Let $\mu(t)$ be an integrating factor for $y' + p(t)y = g(t)$. Recall: What is the differential equation you needed to solve to find μ? What is the formula for μ? Now show that the reciprocal of μ, i.e., $1/\mu$, is a solution of the corresponding linear **homogeneous** equation $y' + p(t)y = 0$. Show that actually, for every constant C, C/μ is a solution.
In other words, by finding an integrating factor μ, you did indeed solve the linear homogeneous equation, namely the solutions are $C/\mu(t)$.

- The case of constant coefficients is well-known to you already: $y' + ky = 0$: Radioactive decay $y' + ky = 0$ or Malthusian growth $y' - ky = 0$. Indeed, as promised on the previous page, "Solutions to this type of equations will be $y = e^{rt}$, you only have to find r by plugging in." Here, the r is of course $-k$, or k, respectively.

- You haven't seen a first order analog of hwk 1.2#21, because that analog would not be of practical interest.

- The general case of variable coefficients may be difficult for *higher* order, but in 1st order we are lucky, because the linear homogeneous first order ODE $y' + p(t)y = 0$ is separable. The power series methods of Chapter 8 announced for 2nd order, still apply for 1st order, but they are not used: they would be overkill.

When you try to find **some** solution to your 1st order linear **in**homogeneous ODE $y' + p(t)y = g(t)$, physics will often guide you; in practice, finding **some** solution means to find the simplest solution. Find here the physical meaning of these simple solution:

Also for the linear inhomogeneus ODE, you have encountered the case of constant coefficients:

$$y' = k(a - y)\,, \quad \text{equivalently} \quad y' + ky = ka\,.$$

It is Newton's law of cooling, and it also represents fall under linear air resistance. Again the exponentials appear in the general solution.

- But if all you need here is just **some** solution, one such solution is easy to guess in this case, and it has the benefit of being special from a physics point of view: take the constant function $y \equiv a$. It's the equilibrium solution, where nothing changes; in the cooling example, it's when the coffee has room temperature, in the parachute example, it's the speed where air resistance balances gravity.

- We also had a cooling example with the inhomogeneity a sine function. That was example 2 in section 3.3 (p. 109–111): They don't write down the equation clearly in the texbook, but rather just plug the data into the solution formula (shame on them!). If you do write down the equation, as discussed in class, you have $y' + ky = a + b\cos\omega t$ (just with different symbols). And the particular solution they discussed in that problem was the one where the exponential term from the general solution had long decayed: It had the structure $y(t) = ?_1 + ?_2 \cos\omega t + ?_3 \sin\omega t$, where each '?' stands for a certain number that depends on the given parameters. When we did the problem, the values for each '?' just popped out of the calculation at the end. Educated guessing would be (a) to guess exactly this formula, but with the '?' really unknown. Then to plug this formula into the equation and determine for what values of the '?' you get indeed a solution.

- We'll discuss what variation of parameters does for first order linear ODEs, when we come to that method. You haven't seen it yet.

The **Superposition Principle** is the basic principle that is characteristic for linear equations. This principle is responsible that the task of solving linear ODEs can be split into two parts as outlined above. It comes in two variants that you should put together in your mind, even though they are separated by several chapters in the book:

Linear combinations: Same notion as in Linear Algebra. There the y_i were vectors, here they are functions. And yes, in Linear Algebra, they have a generalized notion of vectors that encompasses functions as a special case. And this analogy is the very reason why they have such a general concept there.

Thm: *If the functions y_1 and y_2 are solutions to a linear homogeneous (OD)E, then any linear combination $c_1y_1 + c_2y_2$ (with c_1, c_2 constants) is also a solution.* (You find this on p. 161 in the book.)

Thm: *If y_h is a solution to the linear homogeneous ODE*

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

and y_i is a solution to the linear inhomogeneous ODE (note: same stuff on the left, only change is the inhomogeneity!)

$$y'' + p(t)y' + q(t)y = g(t) \tag{3}$$

then $y_h + y_i$ is also a solution to the linear inhomogeneous equation (3). Conversely, if you have found some particular solution y_p to the inhomogeneous equation (3), by whatever method, then any other solution y_i to (3) differs from y_p by a solution of the homogeneous equation (2). (You find this on p. 187 in the book, somewhat differently worded.)

Note that these theorems hold for linear ODEs of any order.

Homework N4#2: Take the linear inhomogeneous equation (3a) $Mv' = -9.81M - kv$, with given numbers k and M, and write down its general solution v_i. Write down the general solution v_h to the corresponding linear homogeneous equation (2a) $Mv' = -kv$ as well. Also find a particular solution v_p to the inhomogeneous equation (3a) by means of physically motivated guessing: Which constant function is a solution to the inhomogeneous equation? Verify the theorem in this concrete example.

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There is one more fundamental ingredient which you need to understand about 2nd order ODEs (and this one has nothing to do with linearity): If you have a 2nd order ODE $y'' = f(t, y, y')$, in order to find a solution y, you must get rid of two derivatives, i.e., you have to do two integrations; this is true at least in principle, even though in practice, it is much more difficult than just integrating twice. Whatever you do in practice to solve the 2nd order ODE (you may not even see the integrations), the two integrations will be responsible that the general solution contains two undetermined constants C_1 and C_2.

Accordingly, the initial value problem for 2nd order ODEs contains two initial conditions: you prescribe, for some t_0 the value $y(t_0) = y_0$ of the function, *and* its derivative $y'(t_0) = y_{0'}$. y_0 and $y_{0'}$ will be given numbers (in Newton's law of motion, they would be initial position and initial velocity, respectively). I have chosen to attach the prime in $y_{0'}$ to the index, rather than using the more common notation y'_0, to make sure $y_{0'}$ looks like a number to you, not like a function.

Remember the Euler method for 1st order IVPs $y' = f(t, y)$, $y(t_0) = y_0$. The ODE permitted us to calculate the rate of change (derivative) of the unknown function y at the initial time t_0 by merely plugging in: $y'(t_0) = f(t_0, y_0)$, and from this rate of change of y, we can get a good approximate value of y at a short 'time' h later:

$$y(t_0 + h) \approx y_0 + hy'(t_0) = y_0 + hf(t_0, y_0) \ .$$

For 2nd order $y'' = f(t, y, y')$, we can similarly get the 2nd derivative (derivative of the derivative, or, rate of change of the derivative) by plugging in: $y''(t_0) = f(t_0, y_0, y_{0'})$, and this information helps us to approximate not only y at a short time h later, but y' as well:

$$\begin{aligned} y(t_0 + h) &\approx y_0 + hy'(t_0) = y_0 + hy_{0'} \\ y'(t_0 + h) &\approx y_{0'} + hy''(t_0) = y_{0'} + hf(t_0, y_0, y_{0'}) \end{aligned}$$

— As for 1st order equations, the IVP for 2nd order equations has a unique solution, under mild hypotheses. We specify the details only for linear equations:[2]

[2] The theorem for nonlinear equations is no more complicated that in the 1st order case, but the notation needed to write it down may cause confusion; this is the only reason why I omit it

Thm: *If p and q and g are continuous in a neighborhood of t_0, then the IVP*

$$y'' + p(t)y' + q(t)y = g(t) \ , \quad y(t_0) = y_0 \ , \quad y'(t_0) = y_{0'}$$

has exactly one solution. The maximal interval of existence of this solution is (at least) as large as the largest interval containing t_0 on which p, q, and g are continuous.

This, too, holds for higher order linear ODEs, with obvious modifications. The last sentence, which specifies how large the interval of existence is, has no analog for nonlinear equations. In nonlinear ODEs, the solution may exist only on a short interval that could not be anticipated from looking at the equation alone. You have seen this phenomenon for 1st order already: reread the middle of p. 2 of my notes on Chapter 1, if you don't remember.

Hwk: N4#3: Determine the maximal interval of existence of the solution to the IVP

$$ty'' - y' + \frac{1}{t^2 - 25}y = e^t \ , \quad y(1) = 1 \ , \quad y'(1) = 17$$

(Careful, don't overlook a little detail.)

N4#4: We'll soon need complex numbers. To make sure you can calculate with them, do the following two (If you have difficulties with these, alert me immediately):

(a) Evaluate $1 + z + \frac{z^2}{2}$ for $z = 1 + i$ (b) Evaluate $\frac{2}{3+2i} + \frac{2}{3-2i}$

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We now study the task of finding *all* solutions to a linear homogeneous equation (program of left column on p. 2 of these notes): Consider three examples:

Example 1: (from hwk 1.2#20a - constant coefficients) – Confronted with the ODE $y'' + 6y' + 5y = 0$, Ann has the wise hindsight / idea to try $y = e^{rt}$ and see if for any r that is a solution. She comes up with two solutions: $y_1(t) = e^{-t}$ and $y_2(t) = e^{-5t}$.

Example 2: (from hwk 1.2#21a - a certain, particularly convenient type of variable coefficients) – Confronted with the ODE $3t^2y'' + 11ty' - 3y = 0$, Bob has the wise hindsight / idea to try $y = t^r$ and see if for any r that is a solution. He comes up with two solutions: $y_1(t) = t^{-1/3}$ and $y_2(t) = t^{-3}$. Bob also notices that $y_3(t) \equiv 0$ is another solution.

Example 3: (new) – Confronted with the ODE $t^2y'' - ty' + y = 0$, Charles has the wise hindsight / idea to try $y = rt$ and see if for any r that is a solution. He comes up with infinitely many solutions: For every r, $y_r(t) = rt$ is a solution.

Who has made more / most progress towards finding all solutions of his/her ODE?

If you merely count solutions, you think Charles is the winner, because he has infinitely many solutions, whereas Ann has two and Bob has three. However, I want to convince you that Ann and Bob are tied for a win, and Charles gets the third place. The reason is that, without extra work, we can immediately create infinitely many solutions from Ann's two solutions, namely $y = c_1y_1 + c_2y_2 = c_1e^{-t} + c_2e^{-5t}$, and that this collection of solutions comprises *all* solutions of Ann's ODE. The same thing applies to Bob's solution, and his extra solution $y_3 = 0$ isn't even needed nor does it contribute. It is redundant, because we get y_3 for free out of y_1 and y_2: namely, $y_3 = 0 = 0 \cdot y_1 + 0 \cdot y_2$.

Charles's infinitely many solutions is just what you get "for free" from a single one, namely $y_1 = t$. His set of solutions falls short of being all solutions. For instance, $y(t) = t \ln t$ is another solution, which Charles hasn't even come close to finding.

It is therefore our next goal to study when two solutions y_1 and y_2 to a linear homogeneous 2nd order ODE are such that their linear combinations $c_1y_1 + c_2y_2$ already make up *all* solutions of the ODE. The answer to this question can be given very quickly (for 2nd order): It must NOT be the case that one solution is a multiple of the other. But you won't take my word for it: think of the implications of this statement: I am claiming that if you have found two solutions of a linear 2nd order homogeneous ODE, subject only to the condition that one shouldn't be a multiple of the other, then there couldn't be any other solution but those that are linear combinations of the two solutions you happen to have already!! How could I make such a bold claim, without even looking at the specific lin'hom'2nd order ODE, nor at your specific solutions?

Remember the margin note from pg 12?

I'll do this argument parallel, for the general case, and for the special case of Bob's example. The special case is meant to show you immediately what is given and what is to be determined.

In its latest edition, the book does this part of the discussion for constant coefficient ODEs only. But as a matter of fact, it works alike for non-constant coefficients; only finding two solutions is not so easy in this case.

Let y_1 and y_2 be solutions to any given lin'hom' 2nd order ODE

$$y'' + p(t)y' + q(t)y = 0$$

In order that any solution y of this equation be expressible in the form $y = c_1y_1 + c_2y_2$, it is certainly necessary that for any given initial values $y(t_0) = y_0$, $y'(t_0) = y_{0'}$, the solution to these initial values is so expressible; in other words, given any $t_0, y_0, y_{0'}$, we must be able to find c_1, c_2 such that

$$\begin{aligned} y(t_0) &= c_1y_1(t_0) + c_2y_2(t_0) = y_0 \\ y'(t_0) &= c_1y_1'(t_0) + c_2y_2'(t_0) = y_{0'} \end{aligned}$$

Let $y_1 = t^{-1/3}$ and $y_2 = t^{-3}$ be Bob's two solutions to the lin'hom'2nd order ODE

$$y'' + \frac{11}{3t}y' - \frac{1}{t^2}y = 0$$

In order that any solution y of this equation be expressible in the form $y = c_1t^{-1/3} + c_2t^{-3}$, it is certainly necessary that for any given initial values, e.g., $y(1) = 5$, $y'(1) = 3$, the solution to these initial values is so expressible; in other words, we must be able to find c_1, c_2 such that

$$\begin{aligned} y(1) &= 5 = c_1 1^{-1/3} + c_2 1^{-3} = c_1 + c_2 \\ y'(1) &= 3 = c_1\tfrac{-1}{3}1^{-4/3} + c_2(-3)1^{-4} = \tfrac{-1}{3}c_1 - 3c_2 \end{aligned}$$

To solve these equations for c_1, c_2, if possible, the following calculation does the job in an organized way: Get c_1 by subtracting $y_2(t_0)$ times the second equation from $y_2'(t_0)$ times the first equation. (The c_2 terms cancel in this procedure.) You get

$$c_1\Big(y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)\Big) = y_0y_2'(t_0) - y_{0'}y_2(t_0)$$

and similarly, subtracting $y_1'(t_0)$ times the first from $y_1(t_0)$ times the second equation,

$$c_2\Big(y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)\Big) = y_{0'}y_1(t_0) - y_0y_1'(t_0)$$

These equations can be solved for c_1, c_2, if the stuff in the parenthesis is different from 0 (and in that case, plugging into the original equations confirms that we have indeed a solution). We still need to show conversely that, if the stuff in the parenthesis does vanish, there are indeed right hand sides y_0, $y_{0'}$ for which we cannot find solutions c_1, c_2. I'll omit the (straightforward, but a bit lengthy) discussion of these cases. If you have studied the theory of systems of linear equations, you will be familiar with the result anyway.

Remember that we argued: "If any solution y can be written as a linear combination $y = c_1y_1 + c_2y_2$ of the two given solutions y_1, y_2, then this must be true *in particular* for the solution for any given pair of initial conditions". Checking this latter condition reduced the problem to a system of linear (algebraic) equations. But as a matter of fact, the '*particular*' above represents indeed the full generality: If you wonder if some particular solution y_h to the lin'hom 2nd order ODE can be represented as linear combination $c_1y_1 + c_2y_2$, then you choose your favorite t_0, calculate $y_h(t_0) =: y_0$ and $y_h'(t_0) =: y_{0'}$ and reason: "We have just determined how to write the solution with initial conditions y_0, $y_{0'}$ as a linear combination. We know that there is exactly one solution to these initial conditions, so this one solution is indeed y_h.

Let's give a name to the 'stuff in parenthesis' above:

Definition: *The Wronskian $W[y_1, y_2]$ of two solutions of a 2nd order linear homogeneous ODE is the function given by the formula*

$$W[y_1, y_2](t) := y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

If you know about determinants, you will recognize the second equality sign as justified. If you do not know about determinants, take the second equality sign as a definition of the rightmost term and be advised that this rightmost term is called a determinant.

Definition: *Let p and q be continuous in some interval I. We call a pair of solutions $\{y_1, y_2\}$ to $y'' + p(t)y' + q(t)y = 0$ a fundamental solution set for this ODE (in the interval I), if $W[y_1, y_2](t_0) \neq 0$ for* **some** *$t_0 \in I$.*

Theorem: *In this case, $W[y_1, y_2](t) \neq 0$ for* **every** *$t \in I$. Every solution of the ODE can be written as a linear combination of y_1 and y_2, if and only if $\{y_1, y_2\}$ is a fundamental solution set.*
Definition: *Instead of saying "$\{y_1, y_2\}$ is a fundamental solution set", we also say that the solution set $\{y_1, y_2\}$ is linearly independent, or that the solutions y_1 and y_2 are linearly independent.*

In Linear Algebra, they would say 'Basis of the solution space' instead of fundamental solution set: Same thing, but historically different names. The notion of linear independence is the same here as in Linear Algebra.

All the arguments about fundamental sets of solutions carry over to higher order lin'hom' ODEs. A set of n solutions $\{y_1, \dots y_n\}$ to a lin'hom' n^{th} order ODE is a fundamental system, if the Wronskian

$$W[y_1, \dots y_n](t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

doesn't vanish. (Note that higher derivatives occur; just as many as you need for specifying enough initial conditions.) This formula involves an $n \times n$ determinant; if you know how to evaluate such determinants, you can check solution sets for higher order ODEs for being a fundamental system. If you have never learned about higher order determinants (which is quite likely unless you have covered them in matrix algebra already), consider this equation as a shorthand for a messy formula, which I don't write down, because you won't need it in this class. [But I do want you to take knowledge just of the plain fact that such a generalization to higher order ODEs is available.]

Finally: How do we find solutions?

After these preparations, we can reveal quite swiftly the available techniques to find solutions:

Alas there is no general technique for all lin'hom' 2nd order ODEs; however, for the very important case of constant coefficients, there is a straightforward technique:

Thm: *For linear homogeneous n^{th} order ODEs with* **constant** *coefficients, the following routine procedure will always find a fundamental set of solutions: Plug in $y = e^{rt}$ and determine r from the (n^{th} degree algebraic) equation ensuing.*
(a) If there are n different real solutions r, then the corresponding functions $y = e^{rt}$ form a fundamental set of solutions.
(b) other cases to be discussed soon.

Hwk: Find the solution to the IVP

$$y'' - (2 + h)y' + y = 0\ , \quad y(0) = 1\ , \ y'(0) = 3$$

for any given $h > 0$.

Let us continue the discussion of how to find solutions for constant coefficient equations. For an n^{th} order lin'hom' ODE with constant coefficients, the educated guess $y = e^{rt}$ leads to an algebraic equation: we have to find the zeros of an n^{th} degree polynomial in the variable r. If we are lucky (case (a) above), this auxiliary equation has n different real solutions. What else could happen? Essentially, two things: (b) We can have complex solutions r of the auxiliary equation, (c) we can have multiple solutions. For sufficiently high order these problems can occur together (multiple complex solutions), but for 2nd order ODEs, we get r from a quadratic equation, so there are only two solutions: they are (a) either both real and different, or (b) both complex conjugates, or (c) one double solution (real).

We deal with these problems in turn.

Example: $y'' - 4y' + 5y = 0$. Try $y = e^{rt}$, get the auxiliary equation $r^2 - 4r + 5 = 0$. The quadratic formula yields the two complex solutions $r_{1/2} = 2 \pm i$.
• The good news is: The general solution is still $y = c_1 e^{(2+i)t} + c_2 e^{(2-i)t}$, according to the same principle as for real solutions r.
• The bad news is: You probably have no idea what e to some complex power actually means, and therefore the previous result is probably (?) meaningless.
• The other good news is that you will learn to make sense out of a formula like $y = c_1 e^{(2+i)t} + c_2 e^{(2-i)t}$. A little algebra will make the complex numbers disappear from this formula again, and then you get a (manifestly) real solution.

Actually, you can check that for this same example $y'' - 4y' + 5y = 0$, there is another fundamental set of solutions that does not use complex numbers: Namely $y_1 = e^{2t}\cos t$, $y_2 = e^{2t}\sin t$. How does this relate to the strange fundamental set of solutions I proposed above, namely $y_+ = e^{(2+i)t}$, $y_- = e^{(2-i)t}$?

The way to make sense of e^z for complex z is by using the Taylor series of the exponential function:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$$

You know this formula for real z already. But the right hand side makes sense for complex z as well; so this series now *defines* e^z for all complex z. (It can indeed be shown that the series converges for all complex z.) And what is even better, all the nice algebra rules like $e^{a+b} = e^a e^b$ remain true, even if a, b are complex. Using them, you can slightly rewrite y_+ and y_-: $y_+ = e^{2t}e^{it}$, $y_- = e^{2t}e^{-it}$. So the e^{2t} that comes up in the formula for y_1, y_2 is already appearing. We are left with understanding e^{it} and e^{-it}. These fellows seem to have something to do with trig functions!!! You are about to witness a miracle:

Once you do complex numbers, you find that the exponential function and trig functions are closely related!

The precise nature of this relationship can again be found by means of power series: Using the power series for the exponential function, you get

$$\begin{aligned}
e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \ldots + \frac{(it)^n}{n!} + \ldots \\
&= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} + \ldots + \frac{(it)^n}{n!} + \ldots \\
&= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + \ldots + (-1)^k \frac{t^{2k}}{(2k)!} + \ldots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - + \ldots + (-1)^k \frac{t^{2k+1}}{(2k+1)!} + \ldots\right) \\
&= \cos t + i \sin t
\end{aligned}$$

This formula is so important that, if the phone rings at 3am and someone asks you "e^{it}?", you are expected to reply "$\cos t + i\sin t$" still half asleep, turn over and continue sleeping as if nothing had happened.

Similarly, you get $e^{-it} = \cos t - i\sin t$, and consequantly $\cos t = \frac{1}{2}(e^{it} + e^{-it})$, $\sin t = \frac{1}{2i}(e^{it} - e^{-it})$.

Conclusion: If the auxiliary equation has a pair of complex solutions $r_\pm = a \pm bi$, then you get corresponding solutions of the lin'hom ODE with constant coefficients $y_1 = e^{at}\cos bt$, $y_2 = e^{at}\sin bt$, because

$$\begin{aligned}
c_+e^{(a+bi)t} + c_-e^{(a-bi)t} &= e^{at}\left(c_+e^{i\,bt} + c_-e^{-i\,bt}\right) \\
&= e^{at}\left(c_+(\cos bt + i\sin bt) + c_-(\cos bt - i\sin bt)\right) \\
&= e^{at}\left((c_+ + c_-)\cos bt + i(c_+ - c_-)\sin bt\right) \\
&= c_1e^{at}\cos bt + c_2e^{at}\sin bt
\end{aligned}$$

Note: When you learned calculus, complex numbers were typically excluded. The earliest reason for this exclusion is the difficulty to graph functions of a complex variable. Much of calculus tolerates complex numbers with indifference. For some parts (roots, logarithms, integrals) complex numbers would cause difficulties inappropriate for 1st year level. Power series in contrast don't merely tolerate complex numbers, they actually beg for them: You have just seen some of the goodies you get from admitting them. What you need to learn about complex numbers in calculus at this level is the following: Basic algebra rules as well as rules for the derivative (product rule, chain rule, derivatives of basic functions) remain true even if complex numbers are involved. And of course you need the relation between trigs and the exponential. However, you avoid logarithms of complex numbers until you understand the pitfalls they pose (which requires some of M 443).

We are left with one more difficulty: what happens if the auxiliary equation has repeated roots? Example: $y'' - 2y' + y = 0$ leads to the auxiliary equation $r^2 - 2r + 1 = 0$ which has the repeated solution $r = 1$. This is the case $h = 0$ of hwk problem N4#5. (The homework assumed $h > 0$.) I'll explain the method later, but will give the result first, as part (c) of the now completed theorem:

Thm: (completion from page 7)
For linear homogeneous n^{th} order ODEs with **constant** *(real) coefficients, the following routine procedure will always find a fundamental set of solutions: Plug in $y = e^{rt}$ and determine r from the (n^{th} degree*

algebraic) equation ensuing.
(a) If there are n different real roots r, then the corresponding functions $y = e^{rt}$ form a fundamental set of solutions.
(b) If some of the roots r are complex, these will automatically come up in pairs of complex conjugates $r = a \pm bi$. To them, there corresponds the pair of solutions $e^{at} \cos bt$, $e^{at} \sin bt$ of the ODE.
(c) If some root r is repeated (double root), solutions for the ODE are e^{rt} and te^{rt}. If the same r occurs with higher multiplicity than two (say k-fold), then k solutions for the ODE are e^{rt}, te^{rt}, ..., $t^{k-1}e^{rt}$.
The set of solutions found according to the above procedure is a fundamental set.

This merits an example that contains all bells and whistles at the same time. Alas 2nd order is too small to attach so many bells and whistles. The examples you encounter will be much simpler, but the complicated example given here is more helpful to understand the theorem:

An ODE of 14th order:

$$y^{(14)}+y^{(13)}+48y^{(12)}+42y^{(11)}+761y^{(10)}+501y^{(9)}+3750y^{(8)}+496y^{(7)}-5856y^{(6)}-8448y^{(5)}-36352y^{(4)}+86016y'''-40960y'' = 0$$

This leads to the auxiliary equation:

$$r^{14}+r^{13}+48r^{12}+42r^{11}+761r^{10}+501r^{9}+3750r^{8}+496r^{7}-5856r^{6}-8448r^{5}-36352r^{4}+86016r^{3}-40960r^{2} = 0$$

Of course you are not expected to find the roots of this mess; however, a symbolic algebra package like Maple or Mathematica will discover that the whole mess can be factored nicely:

$$(r+2)r^2(r-1)^3(r^2+2r+5)(r^2+16)^3 = 0$$

So here is how you get a fundamental set of 14 solutions to the ODE:

factor	contributes	corresponding solutions to ODE	
$r+2$	single root $r=-2$	$y_1 = e^{-2t}$	
r^2	double root $r=0$	$y_2 = e^{0t} = 1$, $y_3 = te^{0t} = t$	
$(r-1)^3$	triple root $r=1$	$y_4 = e^t$, $y_5 = te^t$, $y_6 = t^2e^t$	
(r^2+2r+5)	complex roots $r=-1\pm 2i$	$y_7 = e^{-t}\cos 2t$,	$y_8 = e^{-t}\sin 2t$
$(r^2+16)^3$	complex roots $r=\pm 4i$, each with multiplicity 3	$y_9 = e^{0t}\cos 4t = \cos 4t$, $y_{11} = t\cos 4t$, $y_{13} = t^2\cos 4t$,	$y_{10} = e^{0t}\sin 4t = \sin 4t$, $y_{12} = t\sin 4t$, $y_{14} = t^2\sin 4t$

Let me briefly explain why part (c) of the theorem is true; in other words, why, if there is a double root r, te^{rt} is a solution, in addition to e^{rt}. There is a number of explanations. The naivest explanation is just: plug in the alleged solution and see that it works. But here is a simple argument: If r_0 and $r_0+\varepsilon$ are *different* roots of the auxiliary equation (but very close to each other), we have the solutions $y = c_-e^{r_0t} + c_+e^{(r_0+\varepsilon)t}$ for any choice of constants c_+ and c_-. If r_0 and $r_0+\varepsilon$ are the only solutions to the auxiliary equation (i.e., if we have a second order equation), then we are actually talking about the ODE $y'' - (2r_0+\varepsilon)y' + r_0(r_0+\varepsilon)y = 0$ here. But the following argument does not rely on such an assumption.

In particular, we may choose $c_+ = 1/\varepsilon$ and $c_- = -1/\varepsilon$. This gives the solution $y_\varepsilon := (e^{(r_0+\varepsilon)t} - e^{r_0t}/\varepsilon$. We have chosen the constants in such a way as to find a solution that has a limit as $\varepsilon \to 0$. It is reasonable to believe, and can be proved formally, that the limit of the solution y_ε as $\varepsilon \to 0$ is a solution to the equation with *double* root r_0. (In the absence of further roots, this would refer to the ODE $y'' - 2r_0y' + r_0^2y = 0$.) If you remember the limit definition of the derivative, you observe indeed that

$$\lim_{\varepsilon\to 0} y_\varepsilon = \lim_{\varepsilon\to 0} \frac{e^{(r_0+\varepsilon)t} - e^{r_0t}}{\varepsilon} = \frac{d}{dr}\bigg|_{r=r_0} e^{rt} = te^{r_0t}$$

Basically the same idea, only more elaborate, explains why $t^2e^{r_0t}$ is another solution in case r_0 is a triple root of the auxiliary equation.

You may wish to observe the same phenomenon again, in the hwk I assigned you above (N4#5). You had to find the solution to the IVP

$$y'' - (2+h)y' + y = 0\ , \quad y(0) = 1\ ,\ y'(0) = 3$$

The solution is a bit messy, and you may have sorted the terms together differently. In any case, one way of writing the required solution is:

$$y = \frac{1}{2}e^{(1+\frac{h}{2})t}\left\{\left(e^{\sqrt{\cdots}\,t} + e^{-\sqrt{\cdots}\,t}\right) + \frac{2-\frac{h}{2}}{\sqrt{\cdots}}\left(e^{\sqrt{\cdots}\,t} - e^{-\sqrt{\cdots}\,t}\right)\right\}$$

with $\sqrt{\cdots}$ a shorthand for $\sqrt{h + \frac{h^2}{4}}$. You need l'Hôpital (or the series expansion technique) to carry out $\lim_{h\to 0}$ in the second term of the sum, whereas the limit is straightforward for all subexpressions in the first term of the sum. The limit is $y = \frac{1}{2}e^t\{2+4t\} = t + 2te^t$. This is indeed the solution of the IVP

$$y'' - 2y' + y = 0\ , \quad y(0) = 1\ ,\ y'(0) = 3$$

as you can easily check by plugging in.

Take a deep breath and look back to page 13 what we have accomplished:

We have discussed the superposition principle, which first explains why the task of finding all solutions to a linear inhomogeneous ODE splits into two tasks as outlined on page 13. We have studied how the task of finding all solutions to a linear homogeneous ODE can be accomplished in the case of constant coefficients. There again, the superposition principle played a key role, because the method of educated guessing ("maybe some e^{rt} is a solution") could produce only a few solutions, and the superposition principle spawns all the other solutions out of these. You are already prepared for some review problems.

We next switch over to the right column of page 13, where we want to find just **some** solution of the inhomogeneous equation. Again, the method of educated guessing works wonders in some important cases. But we will also need some more sophisticated techniques to deal with the less simple cases. We start with the case of constant coefficients again:

The method of educated guessing for a particular solution y_p of the linear inhomogeneous equation with constant coefficients is outlined in the book, under the name "Method of undetermined coefficients". It goes as follows: If the right hand side $g(t)$ is an exponential function, you "guess" a constant multiple of that same exponential function as a particular solution. There are some ramifications: If the rhs is a trig function (sin or cos only) or a polynomial, then try a similar trig function or a similar polynomial, respectively, as a particular solution. Details to follow. And there is a technique for trouble shooting, because it may not always be quite that easy. But while the overall technique seems so surprisingly successful, you should also be aware of its limitations. If the rhs is something more complicated (as, eg., $\tan t$, $\sqrt{t^2+1}$, $\arcsin t$) don't bother with trying "similar" expressions for solutions. That will usually not work. The scope of the technique is limited, and it deserves mentioning only because this limited scope does cover some of the very important cases.

Let's see our first example: $y'' + 5y' - 14y = e^t$. You guess $y_p = Ae^t$ with yet unknown A, plug it in, and bingo, you find it works for $A = -1/8$. Next, $y'' + 5y' - 14y = 4e^{-3t}$. You guess $y_p = Ae^{-3t}$, and you find it works with $A = -1/5$. If you have a sum of two exponentials on the rhs, as in $y''+5y'-14y = e^t + 4e^{-3t}$, you can now predict, based on the previous two results, that $y_p = -\frac{1}{8}e^t - \frac{1}{5}e^{-3t}$ is a solution.

Now let's see a potential troublespot: $y'' + 5y' - 14y = e^{2t}$; you try $y_p = Ae^{2t}$ of course, but, too bad, you get $0 \times A = 1$, and you won't find an A that satisfies this equation. The reason for the trouble is that e^{2t} is already a solution ot the homogeneous equation $y'' + 5y' - 14y = 0$, because $r = 2$ is a root of its auxiliary equation $r^2 + 5r - 14 = 0$. The trouble shooting advice tells you, that in this case, you should try ate^{2t}, and that this will succeed, unless $r = 2$ happens to be a *double* root of the auxiliary equation (which is not the case in this example). Guess how you would find a particular solution for $y'' - 4y' + 4y = 17e^{2t}$, where $r = 2$ is indeed a double root of the auxiliary equation: Try Ae^{2t} and see the attempt fail. Try Ate^{2t} and see this better attempt fail again. Try At^2e^{2t} and succeed!

So here is the method:

If you want a particular solution for

$$y^{(n)} + q_{n-1}y^{(n-1)} + \ldots + q_1 y' + q_0 y = g(t)$$

with *constant* coefficients q_0, q_1, ..., q_{n-1} and if $g(t) = ae^{rt}$, then a special solution is $y_p = Ae^{rt}$ where you only need to determine A by plugging in. This works, unless e^{rt} solves the homogeneous equation already (i.e., unless r is a root of the auxiliary equation). In this case, you need to know what multiplicity r has, as a root of the auxiliary equation. If it has multiplicity s, then you know already that e^{rt}, te^{rt}, ..., $t^{s-1}e^{rt}$ solve the homogeneous equation; a particular solution for the inhomogeneous equation is found in the form $y_p = At^s e^{rt}$.

If you have a sum of different terms $g_1(t)+g_2(t)+\ldots$ on the right hand side, and if you can find a particular solution y_{pi} for each term $g_i(t)$ separately, then their sum $y_{p1} + y_{p2} + \ldots$ is a particular solution for the right hand side $g_1(t) + g_2(t) + \ldots$. This is a variant of the superposition principle, and we immediately use it to discuss what happens if $g(t)$ involves trig functions (sin and cos):

Example: $y'' + y' + 5y = 20\cos 3t$. You use $20\cos 3t = 10e^{3it} + 10e^{-3it}$ and employ the method for exponential right hand sides (justly unworried about the complex exponent): you need an A_+e^{3it} term (with A_+ yet to be found) to produce the $10e^{3it}$ on the right hand side. And you need an A_-e^{-3it} to produce the $10e^{-3it}$ on the right hand side. You get the equations $A_+[(3i)^2 + (3i) + 5] = 10$ and similarly $A_-[(-3i)^2 + (-3i) + 5] = 10$. A little arithmetic produces $A_+ = 10/(-4+3i) = -\frac{2}{5}(4+3i)$, $A_- = 10/(-4-3i) = -\frac{2}{5}(4-3i)$. This gives you the solution $y_p = -\frac{2}{5}(4+3i)\,e^{3it} - \frac{2}{5}(4-3i)\,e^{-3it}$. Of course you'd be required to rewrite this whole thing as a real expression:

$$\begin{aligned} y_p &= -\frac{2}{5}(4+3i)\,e^{3it} - \frac{2}{5}(4-3i)\,e^{-3it} \\ &= -\frac{2}{5}(4+3i)(\cos 3t + i\sin 3t) - \frac{2}{5}(4-3i)(\cos 3t - i\sin 3t) = -\frac{16}{5}\cos 3t + \frac{12}{5}\sin 3t \end{aligned}$$

The alternative (for those who dislike complex numbers and are willing to pay with extra memorization for it): If you have a right hand side of $a\cos\beta t$, try $y_p = A\cos\beta t + B\sin\beta t$ and determine A, B by plugging in. If you have a right hand side of $a\sin\beta t$, again try $y_p = A\cos\beta t + B\sin\beta t$. That will work, unless $\pm i\beta$ are roots of the auxiliary equation for the homogeneous equation. Then you have to try $At^s\cos\beta t + Bt^s\sin\beta t$ with s the multiplicity of $\pm i\beta$ as a root of the auxiliary equation.

You have the whole collection of possibilities tabulated in the book. Note that in this table, everything is a special case of case (VII). The right hand sides you can handle are no more and no less than:

polynomial $p_n(t)$ (possibly $p_n \equiv 1$)
$\times$ exponential $e^{\alpha t}$ (possibly $\alpha = 0$, reducing the exponential to 1)
$\times$ $\sin\beta t$ or $\cos\beta t$ or a linear combination thereof (possibly $\beta = 0$, reducing the cosine to 1)

and linear combinations of such terms of such terms. The trial function is then

t^s $\times$ exponential $e^{\alpha t}$ (possibly $\alpha = 0$, reducing the exponential to 1)
$\times$ $(P_n(t)\cos\beta t + Q_n(t)\sin\beta t)$
with $P_n(t)$, $Q_n(t)$ polynomials of the same degree as p_n (A, B, if this degree is 0)

where s is the multiplicity of $\alpha \pm i\beta$ as a root of the auxiliary equation for the homogeneous ODE. If $\alpha \pm i\beta$ is not a root of that equation, then $s = 0$. — The book has a number of useful examples. However, they are a bit bashful in really exploiting the relation between trigs and exponentials and relegate the usage of complex numbers in undetermined coefficients to a group project. I will not exercise this kind of restraint and urge you to become familiar with usages of complex numbers and Euler's formula. Doing so is certainly in line with practice in electrical engineering as well as with efficient calculation skills in mathematics. It conceptually unifies two otherwise separate cases. Complex numbers abound in mathematics, are useful in engineering other than electrical as well, and in physics. – They will also help in some partial fraction decompositions to be encountered later this semester.

Let us now look at an example with all bells and whistles attached. Given sufficient time (and encouragement to persevere), you could be required to do such an example. Make sure you understand all steps.

Sample Problem: Solve the IVP

$$y''' - 3y' + 2y = 5\cos t + 9(t^2-1)e^{-2t} + 8e^{-t}\sin 2t\,, \quad y(0) = \frac{8}{15}\,,\; y'(0) = \frac{1}{10}\,,\; y''(0) = -\frac{9}{5}$$

Solution: We first have to get the general solution, then we have to determine the constants from the initial conditions. The problem to find all solutions to this linear inhomogeneous ODE splits into several parts (because of the superposition principle):

(a) Find the general solution y_h for the homogeneous equation $y''' - 3y' + 2y = 0$

(b) Find a particular solution y_{p1} for the inhomogeneous eqn $y''' - 3y' + 2y = 5\cos t$

(c) Find a particular solution y_{p2} for the inhomogeneous eqn $y''' - 3y' + 2y = 9(t^2 - 1)e^{-2t}$

(d) Find a particular solution y_{p3} for the inhomogeneous eqn $y''' - 3y' + 2y = 8e^{-t}\sin 2t$

We address them in turn:

(a) For the homogeneous equation, we try $y_h = e^{rt}$ and get the auxiliary equation $r^3 - 3r + 2 = 0$. Too bad that it is a cubic equation, but there is some hope to guess one solution by eyeballing. Indeed $r = 1$ works. So we can carry out a long division of polynomials and find $r^3 - 3r + 2 = (r-1)(r^2 + r - 2)$. We can now find all roots: $r^3 - 3r + 2 = (r-1)^2(r+2)$.

We conclude: $r = 1$ is a double root, $r = -2$ is a single root.

$$y_h = c_1 e^t + c_2 t e^t + c_3 e^{-2t}$$

(b) In order to find a particular solution of $y''' - 3y' + 2y = 5\cos t = \frac{5}{2}(e^{it} + e^{-it})$,
we have the choice between the real method and the complex method; in either case we notice that $r = \pm i$ is not a root of the auxiliary equation, so $s = 0$ in this case. The real method says: Try $y = A\cos t + B\sin t$ and determine A and B. The complex method says: Try $y = A_+ e^{it} + A_- e^{-it}$ and determine A_+ and A_-. When done, write the (apparently) complex solution in terms of real trigonometric functions; the coefficients thus obtained will automatically turn out real, unless you have made a miscalculation. I'll choose the real method here; it's probably a bit shorter.

Plugging $y = A\cos t + B\sin t$ into $y''' - 3y' + 2y = 5\cos t$ and collecting terms $\cos t$ and $\sin t$ respectively yields

$$(2A - 4B)\cos t + (4A + 2B)\sin t = 5\cos t$$

and therefore we need $2A - 4B = 5$ and $4A + 2B = 0$. Solving these two equations yields: $A = \frac{1}{2}$, $B = -1$.

We conclude:

$$y_{p1} = \frac{1}{2}\cos t - \sin t$$

(c) In order to find a particular solution of $y''' - 3y' + 2y = 9(t^2 - 1)e^{-2t}$,
we note that the $m = -2$ that is relevant due to the exponent in e^{-2t} is already a root of the auxiliary equation in (a), namely a single root ($s = 1$). The polynomial in front of the exponential is of degree 2. So we have to try $y = t^s P_2(t) e^{-2t} = t(at^2 + bt + c)e^{-2t}$ with undetermined coefficients a, b, c. A bit of hard labor to do here:

$$\begin{aligned}
&y = (at^3 + bt^2 + ct)e^{-2t} \\
&y' = (3at^2 + 2bt + c)e^{-2t} - 2(at^3 + bt^2 + ct)e^{-2t} \\
&y'' = (6at + 2b)e^{-2t} - 2\cdot 2(3at^2 + 2bt + c)e^{-2t} + 4(at^3 + bt^2 + ct)e^{-2t} \\
&y''' = 6ae^{-2t} - 3\cdot 2(6at + 2b)e^{-2t} + 3\cdot 4(3at^2 + 2bt + c)e^{-2t} - 8(at^3 + bt^2 + ct)e^{-2t} \\
&\text{Therefore: } y''' - 3y' + 2y = \\
&= (-8a + 6a + 2a)t^3 e^{-2t} + (-8b + 36a + 6b - 9a + 2b)t^2 e^{-2t} \\
&\quad + (-8c + 24b - 36a + 6c - 6b + 2c)te^{-2t} + (12c - 12b + 6a - 3c)e^{-2t} = 9t^2 e^{-2t} - 9e^{-2t}
\end{aligned}$$

So we conclude:

$$27a = 9\,, \quad -36a + 18b = 0\,, \quad 6a - 12b + 9c = -9\,, \text{therefore:} \quad a = \frac{1}{3}\,, \quad b = \frac{2}{3}\,, \quad c = -\frac{1}{3}$$

and

$$y_{p2} = \frac{1}{3}t(t^2 + 2t - 1)e^{-2t}$$

(d) In order to find a particular solution of $y''' - 3y' + 2y = 8e^{-t}\sin 2t = \frac{8}{2i}(e^{(-1+2i)t} - e^{(-1-2i)t})$, we note that the $r = -1 \pm 2i$ that is relevant due to the exponent here is not a root of the auxiliary equation in (a), so here we have $s = 0$ and our trial function is $y = e^{-t}(A\cos 2t + B\sin 2t)$; or else, if you prefer to take the complex approach, you take $y = A_+e^{(-1+2i)t} + A_-e^{(-1-2i)t}$. Either choice is fine; in contract to (b), I choose the complex approach this time. The reason is that it saves me repeated application of the product rule when calculating y'''. I anticipate this will outweigh the extra labor of returning the (apparently) complex result into a manifestly real form. You can do the real trial function and compare.

There always used to be some folks who think if the rhs is $e^{-t}\sin 2t$ they would check whether $r = -1$ is a root. — No way!! Trigs are imaginary exponentials in disguise, so you have to check if $r = -1 \pm 2i$ is a root. I hope this margin note will eliminate this kind of mistake this year.

$$\begin{aligned}
y &= A_+e^{(-1+2i)t} + A_-e^{(-1-2i)t}\\
y' &= A_+(-1+2i)e^{(-1+2i)t} + A_-(-1-2i)e^{(-1-2i)t}\\
y''' &= A_+(-1+2i)^3e^{(-1+2i)t} + A_-(-1-2i)^3e^{(-1-2i)t}\\
y''' - 3y' + 2y &= A_+\Big((-1+2i)^3 - 3(-1+2i) + 2\Big)e^{(-1+2i)t} + A_-\,cc = -4ie^{(-1+2i)t} + 4ie^{(-1-2i)t}
\end{aligned}$$

Note that I have used the lazybones notation cc as a shorthand for "complex conjugate": The stuff behind A_- is the complex conjugate of the stuff behind A_+; it suffices to evaluate one of them, and the other arises by changing all i to $-i$ in the end. A brief piece of arithmetic gives $(-1+2i)^3 - 3(-1+2i) + 2 = 16 - 8i$, so we have

$$A_+(16-8i) = -4i\ , \quad A_-(16+8i) = 4i\ \text{, hence}\ A_+ = (1-2i)/10\ , \quad A_- = (1+2i)/10$$

Therefore

$$\begin{aligned}
y_{p3} &= \frac{1-2i}{10}e^{(-1+2i)t} + \frac{1+2i}{10}e^{(-1-2i)t} = \frac{e^{-t}}{10}\Big((1-2i)(\cos 2t + i\sin 2t) + (1+2i)(\cos 2t - i\sin 2t)\Big)\\
&= \frac{e^{-t}}{10}(2\cos 2t + 4\sin 2t) = \frac{e^{-t}}{5}(\cos 2t + 2\sin 2t)
\end{aligned}$$

We have now found the general solution of the ODE:

$$y = y_h + y_{p1} + y_{p2} + y_{p3} = c_1e^t + c_2te^t + c_3e^{-2t} + \frac{1}{2}\cos t - \sin t + \frac{1}{3}t(t^2+2t-1)e^{-2t} + \frac{e^{-t}}{5}(\cos 2t + 2\sin 2t)$$

We need to determine c_1, c_2, c_3 from the initial values at $t = 0$. Plugging $t = 0$ into the general solution (and its derivatives, which we therefore have to calculate first), we get

$$y' = c_1e^t + c_2(t+1)e^t - 2c_3e^{-2t} - \frac{1}{2}\sin t - \cos t + \frac{1}{3}\left(-2t^3 - t^2 + 6t - 1\right)e^{-2t} + \frac{e^{-t}}{5}(3\cos 2t - 4\sin 2t)$$

$$y'' = c_1e^t + c_2(t+2)e^t + 4c_3e^{-2t} - \frac{1}{2}\cos t + \sin t + \frac{1}{3}\left(4t^3 - 4t^2 - 14t + 8\right)e^{-2t} + \frac{e^{-t}}{5}(-11\cos 2t - 2\sin 2t)$$

$$\begin{aligned}
y(0) &= c_1 + c_3 + \tfrac{1}{2} + \tfrac{1}{5} &&= \tfrac{8}{15}\\
y'(0) &= c_1 + c_2 - 2c_3 - 1 - \tfrac{1}{3} + \tfrac{3}{5} &&= \tfrac{1}{10}\\
y''(0) &= c_1 + 2c_2 + 4c_3 - \tfrac{1}{2} + \tfrac{8}{3} - \tfrac{11}{5} &&= -\tfrac{9}{5}
\end{aligned}$$

These linear equations can be solved by successive elimination and substitution, with the result:

$$c_1 = \frac{7}{30}\ ,\ c_2 = -\frac{1}{5}\ ,\ c_2 = -\frac{2}{5}$$

The method of **Variation of Parameters** is a method to find solutions to a linear inhomogeneous ODE of any order. For first order, it is equivalent to the method via integrating factors we studied earlier; this equivalence is however a bit disguised, and we will exhibit it below. For higher order with constant coefficients, it is mainly used when the right hand side (inhomogeneity) is not among the special cases (i.e., exponentials, and their offspring sin and cos, and polynomials) that can be handled by undetermined coefficients. The method of variation of parameters is *also* applicable in the case of *nonconstant* coefficients. However, it requires knowledge of a fundamental set of solutions to the homogeneous equation, and we have no general method to find such a fundamental system, except in first order, or for constant coefficients. *If* we have a fundamental system for the homogeneous equation, we can handle any inhomogeneity and any order, and the only difficulty that could prevent us from evaluating the solution all the way is an integral, which we may or may not be able to evaluate explicitly.

Let me first introduce the method for 1st order, where it can be motivated by your previous experience. Take for instance the equation

$$y' + (\tan t)y = \sin^2 t \qquad \text{as an example for the general} \qquad y' + p(t)y = g(t)$$

You have seen previously that for 1st order linear ODEs, the reciprocal $1/\mu$ of an integrating factor μ is a solution of the homogeneous equation. In our example, $\mu(t) = \frac{1}{\cos t}$, and $y_h(t) = 1/\mu(t) = \cos t$ is indeed a solution of the homogeneous equation $y' + (\tan t)y = 0$. When we multiply our ODE with the integrating factor, we get

$$(\frac{1}{\cos t}y)' = \frac{1}{\cos t}\sin^2 t \qquad \text{or in the general case} \qquad (\mu y)' = \mu(t)g(t)$$

or

$$y = \cos t \int \frac{\sin^2 t}{\cos t}\,dt \qquad \text{or in the general case} \qquad y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt$$

So, whereas the general solution of the homogeneous equation is $C\mu^{-1}$, solutions of the inhomogeneous equations are $v(t)\mu^{-1}$, with v being given by some integral, namely $v(t) = \int \mu(t)g(t)\,dt$.

As a cookbook recipe, the method of variation of parameters says therefore: Take the homogeneous solution (here $C\mu^{-1} = C\cos t$), replace the constant of integration by an unknown function v, plug the so obtained function into the inhomogeneous equation and you will obtain v'. You just need to integrate to find v and thus a solution to the inhomogeneous equation. The origin of the name for the method is now clear: the constant of integration is referred to as a parameter (a parameter in the general solution of the homogeneous equation). It is changed into a (nonconstant) function, i.e., it is made to vary. Sometimes the method is also called "variation of constants", for the same reason.

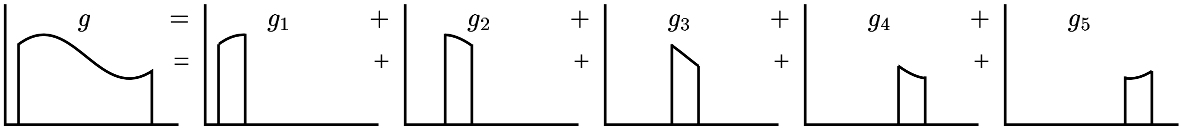
Let's see how this method works on our example $y' + (\tan t)y = \sin^2 t$. First solve the homogeneous equation $y' + (\tan t)y = 0$ (say, by separating variables): You find the general solution of the homogeneous equation $y_h(t) = C\cos t$. Following the method of variation of parameters, you try the substitution $y(t) = v(t)\cos t$ on the inhomogeneous equation; here is what you get:

$$v'(t)\cos t + \underbrace{v(t)(-\sin t) + \tan t\, v(t)\sin t}_{=0} = \sin^2 t$$

The first two terms come from y'. Note that the underbraced terms cancel, precisely because $\cos t$ was a solution of the homogeneous equation, and so you are left with only v', but no v. You get $v' = \frac{\sin^2 t}{\cos t}$, which you can solve by direct integration: $v(t) = \int \frac{\sin^2 t}{\cos t}\,dt = \ln\frac{1+\sin t}{\cos t} - \sin t + C$. The actual evaluation of the integral was of course a quite non-obvious job, and I have deliberately chosen the example so you are prepared for being able or unable to actually evaluate it.

In a moment, we'll see how the method works just as well for 2nd order.

A brief remark: Variation of parameters is another instance of the superposition principle, but in a rather sophisticated way, so that you are not expected to actually see the superposition principle acting here. Take it on good faith from me that using variation of parameters means chopping up the inhomogeneity into a sum of small pulses and using the superposition principle on this sum:



(In contrast to the picture, variation of parameters really chops up the inhomogeneity not in five pulses, but in infinitely many infinitely short pulses, by which a limiting process similar to the Riemann integral is intended. I am not going to elaborate on details how variations of parameters arises from this idea of chopping up the inhomogeneity and using the superposition principle. This would take at least one session, which the syllabus doesn't afford, and is probably not appropriate to work out in an introductory course anyway.)

Let me now exemplify the method in an example of 2nd order:

$$(1-t)y'' + ty' - y = \frac{(1-t)^2}{t}\sin t \quad \text{in the interval} \quad -\infty < t < 1$$

You first need the solutions of the homogeneous equation $(1-t)y'' + ty' - y = 0$. If it were constant coefficients, you could do it routinely; here you are dependent on undeserved luck, or advanced methods. I have particularly prepared the luck for you so I can give you two functions, which you can readily check to be solutions to the homogeneous equation: $y_1(t) = t$, $y_2(t) = e^t$. As you cannot rely on the results about constant coefficients, you should make sure that these two solutions are linearly independent, or, in other words, form a fundamental system of solutions. Their Wronskian is $W = y_1y_2' - y_1'y_2 = (t-1)e^t \neq 0$ in the interval $-\infty < t < 1$.

> Just as an aside, let me note that things would be vastly different, if I had chosen the left hand side to be $(1-t)y'' - ty' - y$ (just one sign change), or $(1-t)y'' - 2ty' + y$. Just so you really see it's a carefully arranged piece of luck, not a routine matter to solve variable coefficients ODEs. — Moreover, I fed *our* example with the simple solution $c_1t + c_2e^t$ to a symbolic algebra package (`Mathematica` Version 4.1), which of course tried its advanced methods, ignorant that I hade fine tuned the example to get *simple* solutions, and it came up with the following useless wisdom:
>
> $$y = c_1e^{t-1}\Gamma(2, t-1) + c_2(t-1)^2\text{Laguerre}_{-1,2}(t-1)$$
>
> involving two functions Γ and Laguerre unknown to you. After insisting with the `Mathematica`ese translation of "are you sure you can't simplify this; try harder to simplify", it came indeed up with $y = c_1t$, so it lost the solution e^t. Seems pretty strange; probably a bug. Just so you see the limitation of these computer packages, marvelous as they are. (Mathematica Version 4.2 gives the same result, but refrains from simplifying the Laguerre term, which is better than a wrong simplification, but short of the useful correct simplification; Maple (Release V) gives the correct result right away. So does `Mathematica` 5.0)
>
> Another thing you may stumble over here is that I told you that the Wronskian of solutions of an ODE either vanishes everywhere or nowhere. Now you have $W = (t-1)e^t$, which vanishes at $t = 1$ and nowhere else. The apparent contradiction is resolved by the fact that the statement about the Wronskian came with the assumption that all coefficients are continuous and that the coefficient of the highest derivative is 1. To ensure the latter, you must rewrite the ODE as $y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = (1-t)\frac{\sin t}{t}$, and you see that $t = 1$ is not permissible. Note that $t = 0$ is also (kind of) a problem; but I have tacitly used the fact that $S(t) := \sin t/t$ can be extended continuously into 0 with the definition $S(0) := 1$.

Returning to our main task after having gone off on tangents a bit, the general solution of the homogeneous equation is $y_h = c_1t + c_2e^t$. We try to find a solution y_p of the inhomogeneous equation; and the variation of parameters principle tells us to try $y_p = v_1(t)t + v_2(t)e^t$, with yet unknown functions v_1 and v_2. At first sight, two problems arise, but at second sight, it turns out that they are each other's solution!

- If we plug $y_p = v_1(t)t + v_2(t)e^t$ into $(1-t)y'' + ty' - y = \frac{(1-t)^2}{t}\sin t$, we get one equation, which is certainly not good enough to determine two unknown functions v_1 and v_2.

• If we plug $y_p = v_1(t)t + v_2(t)e^t$ into $(1-t)y'' + ty' - y = \frac{(1-t)^2}{t}\sin t$, we get an equation involving second derivatives of v_1 and v_2, so it is not clear whether finding v_1 and v_2 is actually easier than the original ODE.

The way that makes these two problems each other's solution is the following: We realize that, whatever the yet unknown solution y_p may turn out to be, there will be different choices of v_1, v_2 representing it: For instance, if y_p should turn out to be (just making this up) t^2, this could happen with $v_1 = t$, $v_2 = 0$, or also with $v_1 = 1$, $v_2 = e^{-t}(t^2 - t)$, or in infinitely many other ways: you choose your favorite v_1, and I can adapt v_2 accordingly. Therefore, we may impose an extra condition on v_1, v_2 that will not affect the solution y_p but will fix v_1 and v_2. This extra condition will be designed just in such a way that second derivatives of v_1 and v_2 cancel. Here's how we do it in practice:

$(1-t)y'' + ty' - y = \frac{(1-t)^2}{t}\sin t$ with

$$y = v_1(t)t + v_2(t)e^t$$
$$y' = v_1'(t)t + v_1(t)\cdot 1 + v_2'(t)e^t + v_2(t)e^t$$

Before taking the next derivative, we impose the extra condition
$v_1'(t)t + v_2'(t)e^t = 0$
which just gets rid of the v_1', v_2' before they have a chance to get differentiated again:

$$y' = v_1(t)\cdot 1 + v_2(t)e^t$$
$$y'' = v_1'(t)\cdot 1 + v_1(t)\cdot 0 + v_2'(t)e^t + v_2(t)e^t$$

If you plug all this into the left hand side, you'll find that all terms $v_{1/2}$ *without* a derivative cancel![3]
$v_1'(t)(1-t) + v_2'(t)(1-t)e^t = \frac{(1-t)^2}{t}\sin t$

The extra condition and the condition obtained from the equation are good enough to determine v_1' and v_2' by merely solving an algebraic system of linear equations:

$$\left.\begin{array}{r} tv_1'(t) + \qquad e^t v_2'(t) = 0 \\ (1-t)v_1'(t) + (1-t)e^t v_2'(t) = \frac{(1-t)^2}{t}\sin t \end{array}\right\} \Longrightarrow \ldots \Longrightarrow \left\{\begin{array}{l} v_1'(t) = \frac{\sin t}{t} \\ v_2'(t) = -e^{-t}\sin t \end{array}\right.$$

Hence

$$v_1(t) = \int \frac{\sin t}{t}\,dt + C_1 \qquad \text{nothing can be done with this integral}$$
$$v_2(t) = -\int e^{-t}\sin t\,dt = \frac{e^{-t}}{2}(\cos t + \sin t) + C_2$$

and the general solution is

$$y = t\left(\int \frac{\sin t}{t}\,dt + C_1\right) + e^t\left(\frac{e^{-t}}{2}(\cos t + \sin t) + C_2\right)$$

You automatically retrieve the contribution of the homogeneous solution from the integration constants in v_1, v_2, even if you were initially modest enough to look for a particular solution only.

Let's do a 3rd order example: The only new thing is that you now have to impose **two** extra conditions, such as to kill the v_j' in every but the last step of differentiation.

$$y''' + 4y'' + 5y' + 2y = \frac{2}{e^t + e^{-t}}$$

The auxiliary equation $r^3 + 4r^2 + 5r + 2 = 0$ has a double root -1 and a single root -2. The general solution of the homogeneous equation $y''' + 4y'' + 5y' + 2y = 0$ is therefore $y_h = c_1e^{-t} + c_2te^{-t} + c_3e^{-2t}$.

[3]This was to be expected: If you first track the undifferentiated $v_{1/2}$ only, you will retrieve exactly such terms *as if* you had erroneously treated v_1 and v_2 as constants; and as $c_1t + c_2e^t$ solves the homogeneous equation, all the undifferentiated $v_{1/2}$ terms cancel after plugging them into the left hand side of our ODE, *as if* the $v_{1/2}$ had been constants.

We therefore try $y = v_1(t)e^{-t} + v_2(t)te^{-t} + v_3(t)e^{-2t}$ for the inhomogeneous equation:

$$
\begin{aligned}
y &= v_1(t)e^{-t} + v_2(t)te^{-t} + v_3(t)e^{-2t} \\
y' &= -v_1(t)e^{-t} + v_2(t)(1-t)e^{-t} - 2v_3(t)e^{-2t} + \overbrace{v_1'(t)e^{-t} + v_2'(t)te^{-t} + v_3'(t)e^{-2t}}^{\text{require this to be } = 0} \\
y'' &= v_1(t)e^{-t} + v_2(t)(t-2)e^{-t} + 4v_3(t)e^{-2t} \quad \overbrace{- v_1'(t)e^{-t} + v_2'(t)(1-t)e^{-t} - 2v_3'(t)e^{-2t}}^{\text{require this to be } = 0} \\
y''' &= -v_1(t)e^{-t} + v_2(t)(3-t)e^{-t} - 8v_3(t)e^{-2t} + v_1'(t)e^{-t} + v_2'(t)(t-2)e^{-t} + 4v_3'(t)e^{-2t} \\
y''' + 4y'' + 5y' + 2y &= v_1'(t)e^{-t} + v_2'(t)(t-2)e^{-t} + 4v_3'(t)e^{-2t} = \frac{2}{e^t + e^{-t}}
\end{aligned}
$$

It's a bit of work to solve these three equations for v_1', v_2', v_3', but it can be done in a straightforward way.

$$
\begin{aligned}
e^{-t}v_1'(t) + \quad te^{-t}v_2'(t) + \ e^{-2t}v_3'(t) &= 0 \\
-e^{-t}v_1'(t) + (1-t)e^{-t}v_2'(t) - 2e^{-2t}v_3'(t) &= 0 \\
e^{-t}v_1'(t) + (t-2)e^{-t}v_2'(t) + 4e^{-2t}v_3'(t) &= \tfrac{2}{e^t+e^{-t}}
\end{aligned}
$$

Add the first and second equation, also add the second and third equation, to eliminate v_1'.

$$
\begin{aligned}
e^{-t}v_2'(t) - \ e^{-2t}v_3'(t) &= 0 \\
-e^{-t}v_2'(t) + 2e^{-2t}v_3'(t) &= \tfrac{2}{e^t+e^{-t}}
\end{aligned}
\qquad \text{and by adding them} \qquad e^{-2t}v_3'(t) = \frac{2}{e^t + e^{-t}}
$$

Substituting back yields

$$
e^{-t}v_2'(t) = \frac{2}{e^t + e^{-t}} \qquad e^{-t}v_1'(t) = \frac{-2(t+1)}{e^t + e^{-t}}
$$

The integrals for v_2 and v_3 are best treated with the substitution $u = e^t$, but alas you won't have much luck with the one for v_1.

$$
v_3 = 2(e^t - \arctan e^t) + C_3 \ , \qquad v_2 = \ln(1 + e^{2t}) + C_2 \ , \qquad v_1 = \int \frac{-2(t+1)e^t}{e^t + e^{-t}}\, dt + C_1
$$

.