$$\int_{-1}^{0} \frac{x\,dx}{x^2+3x+3} = \int_{-1}^{0} \frac{(x+\frac{3}{2})-\frac{3}{2}}{(x+\frac{3}{2})^2+\frac{3}{4}}\,dx = \left[\frac{1}{2}\ln\left((x+\frac{3}{2})^2+\frac{3}{4}\right) - \frac{3/2}{\sqrt{3}/2}\arctan\frac{(x+\frac{3}{2}}{\sqrt{3}/2}\right]_{-1}^{0} = \frac{1}{2}\ln 3 - \frac{\pi}{6}\sqrt{3}$$

(key idea was: complete square and separate fraction to reduce to standard integral)

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^1 \frac{dx}{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}} = \int_{-\pi/2}^{\pi/2} \frac{\frac{1}{2}\cos u\,du}{\frac{1}{2}\cos u} = \pi$$

(key idea was: square root of quadratic polynomial needs trig substitution. complete the square to see which one: namely $x-\frac{1}{2}=\frac{1}{2}\sin u$) — Alternatively, this particular example yields to the "ingenious substitution" $x=\sin^2 t$, which is however NOT a substitution that can be employed routinely for broad classes of integrals.

$$\int x\arctan x\,dx = \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\,\frac{1}{1+x^2}\,dx = \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C$$

(key idea was: integration by parts differentiating $\arctan x$ to get rid of the inverse trig; then PFD for the rational function thus obtained) — Alternatively, you could have tried the substitution $x=\tan u$, but you might find the integration by parts that would still have to follow very unintuitive

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh x} = \int_{-\infty}^{\infty} \frac{2dx}{e^x+e^{-x}} = \int_0^{\infty} \frac{2dt/t}{t+t^{-1}} = 2\int_0^{\infty} \frac{dt}{t^2+1} = \pi$$

(key idea was: rational expressions in e^x are reduced to rational expressions by the substitution $u=e^x$. The resulting integral will need PFD unless it happens to be a standard integral already, which is the lucky case in this example.) — Alternatively you could have used the substitution $v=\tanh(x/2)$ parroting the analog paradigm "rational expression in $\sin x$ and $\cos x$ are dealt with by means of $u=\tan\frac{x}{2}$".

$$\int_0^{\pi} \frac{\cos x}{2+\cos x}\,dx = \int_0^{\infty} \frac{\frac{1-u^2}{1+u^2}}{2+\frac{1-u^2}{1+u^2}}\,\frac{2du}{1+u^2} = \int_0^{\infty} \frac{2(1-u^2)\,du}{(3+u^2)(1+u^2)} = 2\int_0^{\infty}\left(\frac{1}{1+u^2}-\frac{2}{3+u^2}\right)du = \frac{(3-2\sqrt{3})\pi}{3}$$

(key idea is: "rational expression in $\sin x$ and $\cos x$ are dealt with by means of $u=\tan\frac{x}{2}$. Leads to a PFD with arctan type integrals in this case.")

The next integral is the same, but care must be taken with the limits under the substitution $u=\tan\frac{x}{2}$: $x\in\,]0,\pi[$ corresponds to $u\in\,]0,\infty[$; and $x\in\,]\pi,2\pi[$ corresponds to $u\in\,]-\infty,0[$.

$$\int_0^{2\pi} \frac{\cos x}{2+\cos x}\,dx = \left(\int_0^{\infty}+\int_{-\infty}^{0}\right)\frac{\frac{1-u^2}{1+u^2}}{2+\frac{1-u^2}{1+u^2}}\,\frac{2du}{1+u^2} = \ldots = \frac{2}{3}(3-2\sqrt{3})\pi$$

Alternatively, you could have used the periodicity of the integrand to integrate over $[-\pi,pi]$ instead of $[0,2\pi]$, thus bypassing the aforementioned difficulty. But either way, this difficulty should NOT cuase you second thaughts about the $u=\tan\frac{x}{2}$ substitution.

$$\int \frac{(e^x+1)(e^{2x}+1)}{e^{3x}+1}\,dx = \int \frac{(u+1)(u^2+1)}{u^3+1}\,\frac{du}{u} = \int\left(\frac{1}{u}+\frac{1}{(u-\frac{1}{2})^2+\frac{3}{4}}\right)du = \ldots = x + \frac{2}{\sqrt{3}}\arctan\frac{2e^x-1}{\sqrt{3}} + C$$

(key idea is to substitute $u=e^x$, as for all rational expressions in e^x. Here an algebraic cancellation happens to be possible, before or after the substitution.)

$$\int x^3(\ln x)^2\,dx = \frac{x^4}{4}\,(\ln x)^2 - \int \frac{x^4}{4}\,\frac{2\ln x}{x}\,dx = \frac{x^4}{4}\,(\ln x)^2 - \int \frac{x^3}{2}\,\ln x\,dx = \ldots = \frac{x^4}{4}\,(\ln x)^2 - \frac{x^4}{8}\,\ln x + \frac{x^4}{32} + C$$

(key idea is integration by parts, each times differentiating the logarithm(s) until they are gone.)

$$\int_0^{\pi/3} \tan x\,dx = [-\ln|\cos x|]_0^{\pi/3} = \ln 2$$

This may be considered as a standard integral worth memorizing: not the formula, but the fact that the observation $\tan x = \sin x/\cos x$ makes the problem a routine substitution. If you overlook this simple fact, your routine approach would be $u = \tan\frac{x}{2}$ as for all rational expressions in $\sin x$ and $\cos x$, but that makes it overly complicated. – If you continue to overlook that the simple $\sin x$ substitution works here, but invest more smartness towards the cookbook substitutions, you could write $\sin x/\cos x = (\sin x\cos x)/(\cos x)^2$ and substitute $v = \tan x$, which works for all rational expressions depending on "pairs of trigs", namely $\sin x\cos x$, $\sin^2 x$, and $\cos^2 x$ only. That's better, but still more complicated than the given formula. You may wish to try each of them just for exploration purposes. As always, the general-purpose tools are good for many purposes, but for some purposes, they may fall short of being best.

$$\int \frac{x^4+1}{x^3+1}\,dx = \int \left(x + \frac{2/3}{x+1} + \frac{-\frac{2}{3}(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}} \right) dx = \frac{x^2}{2} + \frac{2}{3}\ln|x+1| - \frac{1}{3}\ln(x^2-x+1) + C$$

(A routine PFD problem. For definite integrals, watch when/if $|\cdot|$ is needed, and when you use it, don't fall in the trap if $\int_{-2}^{0}$ is asked for this integrand: it doesn't exist because of $x = -1$, and the $|\cdot|$ in $\ln|x+1|$ prevents the logarithm from warning you with a loud cry 'foul!')

$$\int \exp(-\sqrt{x})\,dx = \int \exp(-u)\,2u\,du = -2(1+\sqrt{x})\exp(-\sqrt{x}) + C$$

Substitution $u = \sqrt{x}$ and one integration by parts, in either order. There is however a paradoxical observation here: If you do parts first, you integrate 1 and differentiate the exponential. If you substitute first, you'll integrate the exponential. But this paradox is psychology, not math. You could have done parts first, integrating $x^{-1/2}\exp -x^{1/2}$ and differentiating the complementary factor $x^{1/2}$, thus going parallel with the integration by parts I did *after* the substitution. But who would *see* such a devious trick beforehand!?

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{(1-\sin t)\,\cos t\,dt}{\cos t} = t + \cos t + C = \arcsin x + \sqrt{1-x^2} + C$$

The moral is: $\sqrt{\text{linear/linear}}$ is as good or as bad as $\sqrt{\text{quadratic}}$. You could have used $\sqrt{\frac{1-x}{1+x}} = \frac{\sqrt{1-x^2}}{1+x}$ instead, by expanding the fraction differently. In either case, a trig substitution is demanded: $x = \sin t$, with $t \in [-\pi/2, \pi/2]$ ($x = \cos t$ with $t \in [0, \pi]$ would have worked just as well.)

$$\int (\arcsin x)^2\,dx = x(\arcsin x)^2 - \int x\,2\arcsin x\,\frac{1}{\sqrt{1-x^2}}\,dx = x(\arcsin x)^2 - 2\int \frac{x}{\sqrt{1-x^2}}\,\arcsin x =$$

$$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\,arcsinx - 2x + C$$

Key idea was: Integrate by parts twice (I've shown one), differentiating the inverse trigs until they're gone.

$$\int_0^t \cos ax\, e^{-bx}\, dx = \ldots = \frac{b}{a^2+b^2} + e^{-bt}\frac{a\sin at - b\cos at}{a^2+b^2}$$

Key idea is either: integrate by parts twice, obtaining the original integral back, and then solve for this yet unknown integral. Or else: write trigs in terms of exponentials and take $e^{(-b+ia)x}/(-b+ia)$ as antiderivative of $e^{(-b+ia)x}$, remembering that complex exponentials are friend, not foe.

$$\int x\sin^2(x^2)\, dx = \frac{x^2}{4} - \frac{\sin 2x^2}{8} + C$$

(A straightforward substitution: $x^2 = u$. Then $\sin^2 u$ can be integrated by the double-angle formula, or through an integration by parts as in problem 41. With parts, you get the same result in a slightly different shape.)

$$\int \sin(\ln x)\, dx = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + C$$

(A straightforward substitution: $\ln x = u$, with an integration by parts following.)

Look at all those hwk problems that tell you about key integrals that cannot be evaluated in terms of elementary functions. Know the Euler Formula and be able to switch from trigs to exponentials and back. Know how to use the $\tan\frac{x}{2}$ substitution. (The trig identities needed will be provided). Be able to select easy and intractable integrals out of a pool of similar-looking problems. I may give problems of the type "Use integration by parts on $\int dx/(1+x^2)$ to evaluate $\int dx/(1+x^2)^2$."