

Homework
UTK – M148 – Honors Calculus II
Spring 2004, Jochen Denzler, MTWF 1:25–2:15, Ayres 309A

1. Use the telescoping trick to calculate $\sum_{j=1}^n j^3$. To save work (and because we only need that much), calculate from $\sum_{j=1}^n j^4$, $\sum_{j=1}^n j^5$ and $\sum_{j=1}^n j^6$ only the highest power in n : (ie., fill in the question marks exactly and give a mere verbal description of what the three dots stand for, in:

$$\sum_{j=1}^n j^4 = ?n^? + \dots, \quad \sum_{j=1}^n j^5 = ?n^? + \dots, \quad \sum_{j=1}^n j^6 = ?n^? + \dots$$

Calculate efficiently: laziness, when bound by good purpose and adorned with prudence, is a virtue...

2. Calculate $\int_0^a x^3 dx$ and $\int_0^a x^6 dx$ as limits of Riemann sums.
 How about $\int_a^b (\frac{71}{23}x^6 + \frac{11}{17}x^3) dx$?
3. Estimate $\int_0^5 \frac{1}{1+x^2} dx$ from above and below by Riemann sums of 10 terms with equidistant nodes.
4. Estimate $\int_0^5 \frac{1}{1+x^2} dx$ from above and below by Riemann sums of 10 terms with nodes chosen in an ad-hoc way, based on a rough sketch of the graph and hindsight, with the purpose in mind to get a more precise result.
5. We'll get a 4-digit precise result for $\int_{-1}^1 \frac{dx}{x^2+1}$ here. You will like a programmable calculator for this, but the main work is still analytic work on the paper. Here is the philosophy of this problem: While Riemann sums are good for theoretical purposes, a practical calculation will give more precise results by using trapezoids rather than rectangles. (We are still using an intuitive, non-rigorous, concept of area and integral here and will not prove that approximation by rectangles and by trapezoids will end up with the same limit.)
- (a) Explain how to save work by only evaluating $\int_0^1 \frac{dx}{x^2+1}$
- (b) Calling $f(x) := 1/(1+x^2)$, find out on which subinterval of $[0, 1]$ we have $f''(x) > 0$ and $f''(x) < 0$ respectively.
- (c) Instead of nesting a small slice of area beneath the graph of f , from x_j to x_{j+1} , between rectangles, nest it between trapezoids: One trapezoid will have the oblique line connecting $(x_j, f(x_j))$ and $(x_{j+1}, f(x_{j+1}))$, the other trapezoid will have the oblique line being a tangent to the graph of f at $x = \frac{1}{2}(x_j + x_{j+1})$. Which trapezoid gives a lower bound for the area, which gives an upper bound, and how does the answer depend on issues discussed previously?
- (d) Write out a formula for the areas of the individual trapezoids involved in the calculation, then sum up the appropriate areas to get an upper and a lower bound for $\int_{-1}^1 \frac{dx}{x^2+1}$: choose 100 subintervals of equal length in $[0, 1]$. Now you may want to use technology and get actual numerical values.

(e) That's the fun part. Look up the appropriate page in the Dictionary of Real Numbers (by Jonathan and Peter Borwein, 1990) and see if you can guess an analytic expression for the numerical value obtained. You may replace the dictionary with your own number experience. (Don't expect this to be possible with just any integral; but here I have concocted a problem for you to be lucky.)

6. Evaluate the following integrals (among the expected answers, there may be: cannot do it with tools available, or, integral does not exist):

$$\begin{array}{ll}
 (a) \int \cos x \, dx & (b) \int (3x^3 - 4 \sin(2x)) \, dx \\
 (c) \int_2^7 \frac{1}{x^2} \, dx & (d) \int_1^5 \frac{\sin x}{x} \, dx \\
 (e) \int_{-2}^2 \frac{1}{x^2} \, dx & (f) \lim_{x \rightarrow \infty} \int_1^x t^{-3/2} \, dt \\
 (g) \int_0^1 x^2 \, dx & (h) \int_0^1 t^2 \, dx
 \end{array}$$

7. Let

$$f(x) := \begin{cases} x^2 & \text{if } |x| < 1 \\ 2|x| - 1 & \text{if } |x| \geq 1 \end{cases}$$

Find $\int_{-1}^3 f(x) \, dx$

8. Take the same function f as before. Calculate $F(x) := \int_0^x f(t) \, dt$. The answer should be a formula that also involves if's. Graph both f and F on the interval $[-2, 2]$ in the same coordinate system.
9. Find the following derivatives without attempting to evaluate the integrals first. The following buzzwords will serve as hints in some of the parts below: set additivity, chain rule. (Next time, you'll need to do similar problems without the buzzword hint).

$$\begin{array}{ll}
 (a) \frac{d}{dx} \int_0^x \sin^2 t \, dt & (b) \frac{d}{dx} \int_x^{50} (\sin^2 t)/t \, dt \\
 (c) \frac{d}{dx} \int_x^{2x} \frac{dt}{1+t^4} & (d) \frac{d}{dx} \int_x^{2x} \frac{dt}{t} \\
 (e) \frac{d}{dx} \int_0^{\sin x} \frac{\sin t}{t^2+1} \, dt & (f) \frac{d}{dx} \int_0^{x^2} g((t+1)^2) \, dt
 \end{array}$$

If one of the results does not qualify for "this answer is surprisingly simple, I can't believe it", look for a mistake, after another glance at the buzzwords.

10. Evaluate $\int_0^3 2x \sin(x^2) \, dx$.
11. Evaluate $\int_{-7}^7 \frac{\sin x}{x^4+1} \, dx$ without even attempting to find an antiderivative. (This would be a doomed attempt.) Graph the integrand, draw your conclusion from what you learn. Can you write your reasoning in formulas as well?
12. Again not bothering about antiderivatives (yet), what does your geometry knowledge tell you should be the result of $\int_{-5}^5 \sqrt{25-x^2} \, dx$? (Graph the integrand.) Try the substitution $x = 5 \sin u$ on this one. Into what other integral does it transform this integral? Mark the quantity u in the graph you have drawn.

- 13.** The Simpson rule is an improved method for the numerical evaluation of integrals $\int f(x) dx$: In the Riemann sum, the integrand f is approximated by constant functions on short intervals. In the trapezoidal and midpoint rule you explored in Problem 5 (albeit without these names given to the method), the function f is approximated with linear functions on short intervals.

Once you realize that a constant function is a polynomial of degree zero, and a linear function is a polynomial of degree 1, you begin suspecting that a polynomial of degree 2, i.e., a quadratic function, may be flexible enough to give a yet better approximation. In this problem, we renounce the intent of obtaining results that are rigorously known to be smaller or larger than the exact integral, but instead just hope for a heuristically good approximation.

For a given function f on a (short) interval $[x_j, x_{j+2}]$, find a quadratic polynomial T_2 that coincides with f at the points x_j , $x_{j+1} := \frac{1}{2}(x_j + x_{j+2})$ and x_{j+2} . Calculate $\int_{x_j}^{x_{j+2}} T_2(x) dx$ exactly, with the result being expressed in terms of $h := x_{j+2} - x_{j+1} = x_{j+1} - x_j$ and $f(x_j)$, $f(x_{j+1})$, $f(x_{j+2})$. Accompany your calculations with a figure and be sure to state your result in a clear, concluding statement that conveys the purpose of the whole calculation together with its result.

- 14.** If you integrate a polynomial of degree 2 or less with the Simpson rule, the result will always be exact, by definition.

Now calculate $\int_0^1 x^3 dx$ (approximately) with the Simpson rule with nodes $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$ and compare with the exact result.

Also calculate $\int_0^1 x^4 dx$ (approximately) with the Simpson rule with nodes $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$ and compare with the exact result.

Finally calculate $\int_0^\pi \sin x dx$ (approximately) with equidistant nodes and *four* (ie., two pairs) intervals only; remember that Simpson's rule uses one pair of intervals at a time. Compare with the exact result. How do you like the precision thus obtained?

- 15.** For this problem, you need to review your trig functions: Suppose $\tan(u/2) = 17/29$. Calculate $\cos u$. I know you can do this with the pocket calculator; and it's ok to use it for checking, but here I want more: I want you to give $\cos u$ as an exact rational number in the form p/q with integers p, q .

Same question for $\sin u$.

It's ok to look up formulas for $\cos 2x$ and $\sin 2x$ in the table and quote them; use your own reasoning for the rest.

In the course of this problem, you'll come up with a formula that expresses $\cos u$ and $\sin u$ each in terms of $\tan(u/2)$ only. Put a color frame around these formulas; they will turn out to be very useful in calculating certain integrals!

- 16.** The functions \sinh and \cosh are defined as follows:

$\cosh x := (e^x + e^{-x})/2$, and $\sinh x := (e^x - e^{-x})/2$. Show:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

- 17.** (Fill up your table of derivatives to accommodate the functions recently discussed).

18. Do the following indefinite integrals:

$$\begin{array}{ll} \int \frac{dx}{x^2 + a^2} & \int \frac{dx}{(x + a)^2} \\ \int \frac{dx}{x + a} & \int \frac{x}{x^2 + a^2} dx \\ \int \frac{x + a}{x^2 + a^2} dx & \int \frac{\ln x}{x} dx \\ \int \frac{1}{x \ln x} dx & \int \ln x dx \end{array}$$

Hint for the last one: For the time being, this requires hindsight: Calculate $\frac{d}{dx}(x \ln x)$ first and then do the problem by eyeballing.

19. Calculate

$$\int \frac{x + 1}{x^2 + 2x + 1} dx \quad \text{and} \quad \int \frac{1}{x^2 + 2x + 1} dx .$$

Then, combining the experience from these two, calculate

$$\int \frac{x}{x^2 + 2x + 1} dx .$$

Calculate

$$\int \frac{x + 1}{x^2 + 2x + 2} dx \quad \text{and} \quad \int \frac{1}{x^2 + 2x + 2} dx .$$

Then, combining the experience from these two, calculate

$$\int \frac{x}{x^2 + 2x + 2} dx .$$

20. Calculate $\int_0^1 \frac{x + 1}{x^2 + x + 1} dx$.

21. Show that $\frac{1}{x(x + 1)} = \frac{1}{x} - \frac{1}{(x + 1)}$. Then calculate $\int \frac{dx}{x(x + 1)}$. Use the same trick to calculate $\int \frac{dx}{x(x + 3)}$.

22. The graph $y = \sin x$ ($0 \leq x \leq \pi$) rotates around the x -axis. Find the volume of the rotation body thus obtained. In order to carry out the integral, you need to use a double angle formula to make the trigonometric integrand amenable for finding an antiderivative.

23. Calculate the arclength of the curve $y = \sin x$ ($0 \leq x \leq \pi$). Do *not* attempt to evaluate analytically the integral obtained. That would be a doomed attempt. Trigs under square roots are always unamenable to analytic integration, except in the cases (a1), (a2) mentioned in Problem 32 below.

24. Calculate the surface of the rotation body obtained by rotating the curve $y = \sin x$ ($0 \leq x \leq \pi$) around the x -axis. Leave the integral unevaluated in this problem. We'll later pursue many variants to do this integral.

- 25.** Calculate the volume and the surface area of a torus: a torus is the donut shaped object which you get when a circle of radius r rotates about a line that is a distance $R > r$ away from the center of the circle. (The line is in the same plane as the circle.)
- Specify also the volume and surface of the “outer half” and “inner half” of the torus separately. For the volume calculation, compare the shell method with the “washer” method.
- 26.** (a) The curves $y = x^2$ and $y = 6 - x^2/2$ enclose an area. Calculate it.
- (b) Suppose this area is rotated around the y -axis. What is the volume of the body thus obtained? Calculate the volume in *two different ways*: shells and discs.
- (c) Calculate the surface area of that same body.
- (d) Now rotate the same area about the line $y = -1$. What volume does this rotation body have?
- 27.** Write the arclength of an ellipse with semi-axes a and b as an integral. The ellipse in question is described by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Do not attempt to evaluate the integral analytically (hopeless as in Pblm 23).
- 28.** A spherical cap of ‘radius’ ϑ (an angle!) is that part of the sphere that is located ‘North’ of the geographical latitude $\frac{\pi}{2} - \vartheta$. A sphere of radius R is of course obtained by rotating the circle $x^2 + z^2 = R^2$ about the z -axis. Design a formula for the area of a spherical cap of radius θ on a sphere of radius R .
- 29.** From a ball of radius R , a cylinder with radius r has been drilled out centrally (i.e., the axis of the cylinder passes through the center of the ball). We intend $r < R$ of course. What is the volume of the remaining body?
- 30.** The graphs $y = c + \frac{1}{a} \cosh ax$ describe the shapes of hanging chains (with homogeneous mass distribution). Calculate the length of this graph between $x = -b$ and $x = b$. Use your result to determine how high above the ground a chain of length 30, suspended between the points $(-10, 20)$ and $(10, 20)$, will hang at its deepest point.
- 31.** If you rotate the graphs of $y = \frac{1}{a} \cosh ax$ about the x axis, you get such shapes as will be taken on by a soap film suspended between two parallel circles: namely the rotational surfaces generated by the above graph between $x = -b$ and $x = b$. (Some of these shapes, depending on a and b , will actually be unstable and thus not be taken on by a real soap film, but others will be stable and can be observed in reality.)
- Calculate the area of the mentioned surface between $x = -b$ and $x = b$.
- 32.** Now we evaluate the integral from Problem 24 in several steps; the motivation is given for each step:
- (a) Trigs under a (square) root make the integral untreatable, except in two mitigating circumstances: (a1) You can get rid of the trigs under the square root by means of a substitution (without introducing inverse trigs), or (a2) You can get rid of the root, because the term beneath the root is actually a complete square in disguise. — With (a1) in mind, substitute either $\cos x = s$ or $\sin x = s$. (Only one of the two works! It’s your job to figure out which one.)
- (b) Since you got rid of the trig under the square root, you call it progress. But we’re not done yet. Now let’s kill the square root by making what is under it a complete

square. Two possible substitutions do it here: (b1) $s = \tan u$ or (b2) $s = \sinh v$. It looks insane to reintroduce trigs (or hyps) after we have just gotten rid of the trigs, but this time we get rid of the square root, which makes it a bargain. Pursue both variants (b1) and (b2)

You call the result progress again, since now the root is gone. We'll get rid of the trigs or hyps in the next problem.

- 33.** For the integral from (b2) in the previous problem, a similar technique works as in the case of $\int \sin^2 x \, dx$. Do it and get a result. (Lucky you: what a nice shortcut! The next paragraph shows the routine way, which is more tedious:)

The integral you get from (b1) in the previous problem is more daunting. Here is the cookbook recipe, which works for all rational expressions in terms of $\cos u$ and $\sin u$ only: It says: *substitute* $t = \tan(u/2)$. That looks strange since $\tan u/2$ can't be seen anywhere in the integral, but that's what works. Use problem 15 and do it. Now you get a rational function. Call it progress, because the trigs that were introduced to kill the square root are now also gone. We'll learn soon how to do integrals of rational functions.

We'll discuss later *why* the $\tan u/2$ substitution is so good (and also, when there are even better possibilities.)

- 34.** Before pursuing our latest integral (of a rational function only), let's do a quick test against possible miscalculations: The following simple and cute fact is worth knowing (and maybe you do know it already), even though it's not part of Calculus 2:

The isoperimetric inequality: Among all bodies with a given volume, the ball alone has the smallest surface area. Among all planar domains with a given area, the disc alone has the smallest perimeter.

Compare the surface area calculated in 33 with the surface area of the ball of the same volume (you calculated the volume in 22.) Does your result satisfy the test against the isoperimetric inequality?

- 35.** In this problem, we have two cylinders: the first cylinder C_1 has the x -axis as its axis of symmetry, and it has radius 1. The second cylinder C_2 has the y -axis as its axis of symmetry; it has radius 1 as well. Draw a sketch.

These two cylinders intersect. And you are to figure out the volume of the intersection body:

As a hint, I ask you to draw to-scale sketches of the cross section at $z = 0$, $z = \pm 1/2$, and $z = 1$. You will obtain the volume as an integral of the form $\int_a^b f(z) \, dz$.

- 36.** The cardioid is a heart-shaped curve that is traced out by a point on the circumference of a circle while this circle rolls along another (fixed) circle of the same size. The cardioid can be described in polar coordinates by the equation $r = 2(1 - \cos \varphi)$. Calculate its length, and the area of its interior. Apply the isoperimetric inequality test (mentioned in Pblm 34) as a simple check against possible miscalculation.

The same curve can also be described in parametric form $x = 2 \cos t - \cos 2t - 1$, $y = 2 \sin t - \sin 2t$. Use this form to get an independent calculation of the arclength. (To use this form of the equation for the calculation of the area could be barely possible here, but is better deferred to multivariable calculus.)

- 37.** The equation $x^{2/3} + y^{2/3} = 1$ gives a diamond shaped curve. Calculate its area, and its perimeter.

If you are careful, you should encounter a formal problem at $x = 0$, when calculating the perimeter. What is the problem? Avoid it by chopping out an interval $[-\varepsilon, \varepsilon] \ni x$ from the curve, and taking the limit $\varepsilon \rightarrow 0$ *after* the integration.

- 38.** Want a sophisticated one? Following up on Problem 35, we now take *three* cylinders of radius 1. Their axes are the x axis, the y axis, and the z axis respectively. What is the volume of the intersection of all three cylinders?

- 39.** This one is on the borderline to multivariable already, but let's treat it here, because the integral is a *very* important one. We are talking about the *definite* integral $I := \int_0^\infty e^{-x^2} dx$.

First of all, I should say that $\int_0^\infty e^{-x^2} dx$ is a shorthand for $\lim_{N \rightarrow \infty} \int_0^N e^{-x^2} dx$. Next I must tell you that you won't be lucky if you try to find a formula for the antiderivative of e^{-x^2} . What we are going to do is to rotate the area between $z = e^{-x^2}$ and the x axis about the z axis and determine the volume of the rotation body thus obtained. The rotation body will be a bell shaped hump sitting on the x - y plane.

(a) Use the shell method to determine the mentioned volume.

(b) Now express one fourth of that same volume (namely the part of the bell shaped hump that is above the positive quadrant $x > 0, y > 0$) in terms of the yet unknown integral I , using a *different* slicing technique: take the area $A(y_0)$ of slices of the body cut by the plane $y = y_0$, express $A(y_0)$ as a multiple of $A(0) = \frac{1}{2}I$, and integrate.

(c) Combining (a) and (b), find I .

- 40.** For every real number $t > 0$, we define the integral

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx := \lim_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty} \int_\varepsilon^N x^{t-1} e^{-x} dx$$

Simple evaluations of this integral are only available for integers and half-integers t . This integral occurs quite frequently and is a good training example for integration by parts:

(a) Suppose you had evaluated $\Gamma(t)$ already, how do you obtain $\Gamma(t+1)$ from it? Find a formula for $\Gamma(n)$ for all positive integers n , step by step beginning at $n = 1$.

(b) Use a substitution to calculate $\Gamma(\frac{1}{2})$ from the result of the previous problem.

(c) What is $\Gamma(\frac{3}{2})$, $\Gamma(\frac{5}{2})$, $\Gamma(\frac{7}{2})$?

- 41.** Use repeated integration by parts to get $I_n := \int_0^{\pi/2} \sin^n x dx$ for $n = 1, 2, 3, 4, \dots$. Find the general principle how to get $\int_0^{\pi/2} \sin^n x dx$ from $\int_0^{\pi/2} \sin^{n-2} x dx$ and write down a general formula for I_n , separately for even n and odd n .

- 42.** Integration by parts may be helpful for preprocessing an integral that needs to be evaluated numerically: Suppose you want to find $\int_1^{100} \frac{\sin x}{x^{1/2}} dx$. With about 30 oscillations and 10 nodes per oscillation, you need to calculate 300 terms for modest precision, and you use this work mainly to calculate positive and negative terms that will cancel out to a significant extent.

Use a few integrations by parts to damp the amplitude of the oscillations.

43. This is another, extra cute one: The integral

$$M(a, b) := \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}$$

with positive numbers a, b cannot be evaluated in formulas (unless $b = a$), but can still be calculated very very rapidly, once you have exhausted the formal manipulation toolkit.

(a) Calculate $M(a, a)$. (That's easy)

(b) Based on (a), evaluate $M(2.7, 2.8)$ with less than 4% error margin in a one-liner with a small number of keystrokes on the pocket calculator.

(c) Here is the heavy work. Use the substitution $u = \frac{1}{2}(x - \frac{ab}{x})$. It's algebraically challenging and doesn't seem to promise anything, but it will show you that $M(a, b) = M(\sqrt{ab}, \frac{a+b}{2})$.

(d) Use the result of (c) repeatedly to evaluate $M(1, 10)$ with a few keystrokes on your pocket calculator.

This is a ping-pong problem. You must write it up in a clean presentation. Scribbling it in some corner etc. will likely introduce miscalculations that can hardly be found.

44. A simple integration with a sophisticated consequence:

You know that for $x > 0$, $\sin x < x$. By integrating this inequality, get a lower bound for $\cos x$: Namely $\cos x \geq ???$. By integrating again, obtain a lower bound for $\sin x$. Continuing, obtain an upper bound for $\cos x$.

List the whole sequence of upper and lower bounds for both $\sin x$ and $\cos x$ that you obtain this way (for $x > 0$).

Using the fact that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, comment on what happens to the previous estimates in the case $x < 0$.

45. You know that $e^{-x} < 1$ for $x > 0$. By integrating once, obtain a lower bound for e^{-x} . Integrating this lower bound again, obtain an upper bound for e^{-x} . Keep going, such as to spawn more upper and lower bounds for e^{-x} , always assuming $x > 0$.

You know that $e^x > 1$ for $x > 0$. What information do you get if you reuse the method of the previous paragraph on this function, for $x > 0$?

Based on this, and the previous problem, we will soon be able to understand a deep relation between trigs and exponentials, in a similar way as we have a relation between the hyperbolic functions and exponentials.

46. Find the partial fraction decomposition of

$$\frac{x^8 + 7x^7 + 18x^6 + 25x^5 + 56x^4 + 4x^3 + 18x^2 - 27x - 66}{(x^2 + 2)^2(x^2 - 1)(x + 3)}$$

- 47. Practical vs. theoretical convergence:** Calculate the first 20 terms of the power series $\sum_{n=0}^{\infty} (-1)^n n! x^n$, list them, and list their partial sums, for $x = 0.05$ and $x = 0.1$. Does the series *seem* to converge (and if so, what could be its approximate limit, to be judged experimentally)? By the ratio test, does the series indeed converge?

Same question for the series $\sum_{n=0}^{\infty} (-1)^n x^n / n!$ with $x = 15$ and $x = 4$.

- 48.** Assume $\sum_{n=0}^{\infty} a_n x^n$ has a positive radius of convergence and the value of the series is $f(x)$. Express $f(0)$, $f'(0)$, $f''(0)$, \dots , $f^{(n)}(0)$ in terms of the coefficients $a_0, a_1, a_2, \dots, a_n$.

- 49.** By explicit multiplication of power series, with

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and using the binomial theorem, verify that $E(x)E(y) = E(x+y)$.

Notes: You may know the binomial theorem only for $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, but not the general formula for $(a+b)^n$. In that case, you could only check the first few terms in the power series formula $E(x)E(y) = E(x+y) = 1$. We'll fill in the rest in class then.

Note that problem 45, together with the convergence of the series, guarantees that $e^x = E(x)$ for $x < 0$, whereas for $x > 0$, that problem only allowed to conclude $e^x \geq E(x)$. But in this problem, you show (in particular) that $E(x)E(-x) = E(0)$. This implies $e^x = E(x)$ also for $x > 0$, once it is established for $x < 0$.

- 50.** Find the power series for $1/(1+t^2)$ from the geometric series. By integration ($\int_0^x \dots dt$), obtain the power series for $\arctan x$. What is the radius of convergence for either series?
- 51.** Using long division of power series (with the series for $\sin x$ and $\cos x$), find the first three terms of the power series of $\tan x$.

Note: To find the full power series for $\tan x$ by this method is well nigh impossible. This power series has a radius of convergence of $\pi/2$. With the tools available, you have no chance in proving that $\pi/2$ is indeed the radius of convergence. However, you *can* answer the following question with the tools you have available:

Show for a power series whose sum is $\tan x$ within its disk of convergence: Its radius of convergence can be *at most* $\pi/2$.

- 52.** Think complex numbers, and ignore that you have already figured out the power series for $1/(1+x^2)$. Pretend you don't know this power series, or its coefficients, yet.

Show for a power series whose sum is $1/(1+x^2)$ within its disk of convergence: Its radius of convergence can be *at most* 1.

- 53.** Use the addition theorem for \sin and \cos to obtain an expression for $\tan(u+v)$ in terms of $\tan u$ and $\tan v$. From this expression, write $\arctan x + \arctan y$ as \arctan of a *single* expression involving x and y . Building on the formula thus obtained, evaluate $\arctan \frac{1}{5} + \arctan \frac{1}{5}$, then $4 \arctan \frac{1}{5}$, and finally $4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$.

The latter answer should turn out to be $\arctan 1$, which is $\pi/4$. Did you get it? In the early days of calculus, with calculations done by hand, this expression (and the power series for \arctan) was used to get high precision numerical approximations to π .

54. Use power series to determine the following limit:

$$\lim_{x \rightarrow 0} \frac{\ln(1-x) + x - 1 + \cosh x}{\tan x - x}$$

55. Find the radius of convergence of

$$(a) \sum_{n=0}^{\infty} \frac{n!^2}{(2n)!} x^n \quad \text{and} \quad (b) \sum_{n=0}^{\infty} \frac{n!^2}{(2n)!} x^{2n}$$

repectively.

56. An ellipse with eccentricity $\varepsilon < 1$ can be written in the parameter form $x = \cos \varphi$, $y = \sqrt{1 - \varepsilon^2} \sin \varphi$. Write its perimeter as an integral. Unless $\varepsilon = 0$, the integral cannot be evaluated in terms of elementary functions. But evaluate the integrand as a power series in powers of ε . Use the binomial series, integrate it term by term, and remember the earlier problem where we integrated powers of $\sin \varphi$ (or, equivalently, of $\cos \varphi$). You can obtain the *full* power series (i.e., a formula for the n -th term).

57. In this problem we study the power series for $1/(1-x-x^2)$ from various points of view:

(a) Calculate the first three terms ($f_0 + f_1x + f_2x^2 + \dots$) directly from Taylor's formula.

(b) Calculate terms up to (including) order x^4 by long division of power series.

(c) Write the power series with unknown coefficients f_n , namely $(1-x-x^2)^{-1} = \sum_{n=0}^{\infty} f_n x^n$, and obtain a relation among the f_n by multiplying out $1 = (1-x-x^2) \sum_{n=0}^{\infty} f_n x^n$ and equating coefficients. Use this relation to evaluate about a dozen of the f_n in quick succession. Note that this route (closely related to the previous one) tells you that all the f_n are integers.

(d) Plug $y = x + x^2$ into the (geometric) series for $1/(1-y)$ and expand and reorder the terms. (This procedure is legitimate even though we haven't discussed it yet, at the time I am typesetting this.)

(e) Use a partial fraction decomposition of $1/(1-x-x^2)$ to obtain $\sum_{n=0}^{\infty} f_n x^n$ as the sum of two geometric series.

(f) The numbers given by the formulas $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ are called Fibonacci numbers, and are a popular object in recreational mathematics. Use (c) and (e) together to find a 'formula' for the n -th Fibonacci number. Note how this formula hides the fact that Fibonacci numbers are integers.

(g) What is the radius of convergence of the power series $\sum_{n=0}^{\infty} f_n x^n$?

58. Assume $p > 1$. Write down the geometric series for $(1 - \frac{1}{p})^{-1}$. Now take three samples of this geometric series, namely for $p = 2$, $p = 3$, and $p = 5$. Multiply them out explicitly. (Use rationals. Don't even think of using decimals, which would spoil the fun completely.) Order the resulting terms by size, and exercise patience: make sure to calculate at least 15 terms. Can you describe in words the totality of all terms that will arise under multiplication? Is $1/30$ among the terms? Is $1/31$ among them? $1/150$? Why/why not?

59. In antiquity, Euclid reasoned: Suppose, there were only finitely many prime numbers, say p_1, p_2, \dots, p_N . Then $p_1 p_2 \cdots p_N + 1$ cannot be divisible by any of them, so it must

have *other* prime factors or be a new prime itself. This contradiction proves that there must be infinitely many prime numbers.

Euler invented the argument: “Suppose there were only finitely many prime numbers. Then the harmonic series would have to converge, which we know it doesn’t. Therefore there must be infinitely many prime numbers.” — Do you understand this reasoning? Explain.

Without going into details here, just as a background information: By throwing a little bit of extra theory at basically the calculation you have just gone through in this and the previous problem, Euler figured out that $\sum 1/p$, the sum of the reciprocals of all prime numbers, is divergent.

- 60.** Remember Abel’s theorem: If a power series $\sum a_n x^n$ still converges at some point x_* on the boundary of the disc (interval, if you want to stick with real numbers) of convergence $|x| < \rho$; and if moreover the sum $f(x) = \sum a_n x^n$ has a limit $\lim_{x \rightarrow x_*} f(x)$, then the two are the same: $\lim_{x \rightarrow x_*} f(x) = \sum a_n x_*^n$.

(a) What is the value of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1}/n$?

(b) Now we rearrange the series by taking alternately two positive and one negative term: $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$, and you know that this may produce a different result, since the series is only conditionally convergent. We will here proceed to actually evaluate this sum. To this end, we find a function f whose power series is

$$f(x) = \frac{x^1}{1} + \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^8}{4} + \frac{x^9}{9} + \frac{x^{11}}{11} - \frac{x^{12}}{6} + \dots$$

This power series converges for $|x| < 1$, and you may take it apart into two series:

$$f(x) = \sum_{n=?}^{\infty} \frac{x^{?n+?}}{?n+?} - ? \sum_{n=?}^{\infty} \frac{x^{?n+?}}{?n+?}$$

Fill in the blanks (?), think of how the geometric series and the series for $\ln(1-x)$ look and how they are related, and see whether you can write $f(x)$ as the sum of antiderivatives of two easy series.

Now, what is the value of $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots$?

- 61.** An unknown function f , given by a convergent (for sufficiently small x) power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

satisfies the equation

$$f(x) = 1 + \frac{x}{6} (f(x)^3 + 3f(x)f(x^2) + 2f(x^3)) .$$

Calculate $a_0, a_1, a_2, a_3, a_4, a_5, a_6$.

This power series has basically been invented By George Pólya for the sole purpose of studying its coefficients a_i . One couldn’t care less about the function f itself. The surprise is that the series is “counting” isomeres of the alcohols $C_n H_{2n+1} OH$. There are a_n different such isomeres. While we are just cannibalizing his series here for some training in calculations with power series, this problem also serves as an illustration that power series have applications in surprising areas, like eg. combinatorics (the mathematics of counting sophisticated things in a sophisticated way). The previous problems have already indicated that calculus, and series, may be useful to study prime numbers; and actually other number theoretical questions.