— Fourth manuscript on integrals, expanding on 7.4 of our textbook —
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7.4: Trigonometric Substitutions (and Hyperbolic ones, too)

There is nothing special about trigonometric substitutions. What you are going to learn here is not something *new to do*, but rather to *see* certain options which you would not have considered before, i.e., to do something without a hint that you could have done with a hint before.

The textbook covers *only* trigonometric substitutions in its chapter 7.4. It does not include hyperbolic substitutions. I think this is somewhat unlucky, because you can see the coherence of ideas much better, if you consider trig and hyp substitutions together. After all, you will want to exploit the analogy between trig and hyp to organize your memory, so you should see both of them.

Moreover, examples 1 and 3 in the textbook become extremely simple with hyp substitutions, so they are not too convincing a case for the trig substitutions (but for the fact that they hide the alternative from you).

7.4.1: Purpose

One of the basic purposes of trig substitutions is to get rid of square roots of some quadratic polynomial under the integral. E.g., in

$$\int \sqrt{a^2 - x^2}\, dx\ ,$$

you can substitute $x = a\sin\theta$ and get $\int a^2 \cos^2\theta\, d\theta$. Given that there are no trigs in the integral in the very first place, and that you are introducing them by means of the substitution, this may be a surprising approach at first glance. It is the purpose of this lesson to demystify this approach and to convince you that it is a very natural thing to use such a substitution.

You may have use for trig substitutions even in the absence of square roots, as we will see in 7.4.3(d), and moreover, in 7.4.4, you will also learn a trig substitution whose purpose is to *get rid* of trigs (leading to rational functions) rather than to introduce trigs in exchange for square roots. For the moment, however, we will discuss the situation where you bring in trigs in order to kick out square roots.

We will only handle the square roots $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. As usual, you may handle cases like $\sqrt{x^2-x}$ by completing squares: $\sqrt{x^2-x} = \sqrt{(x-\frac{1}{2})^2 - \frac{1}{4}}$, which is of the form $\sqrt{y^2-a^2}$. Moreover, you can make square roots of quotients of linear expressions fit into our scheme, according to the paradigm (this example is for $x > 3$)

$$\sqrt{\tfrac{x-1}{x-3}} = \tfrac{\sqrt{(x-1)(x-3)}}{x-3} = \tfrac{\sqrt{(x-2)^2-1}}{x-3}\ .$$

We will NOT handle square roots of polynomials of degree higher than 2, nor cube roots, nor situations where different square roots of quadratic polynomials occur, like in $\sqrt{x^2+1}/\sqrt{x^2-1}$. You should consider such cases as vastly different (i.e., not at all $\int$imilar!) from the cases considered here, and you will typically not touch them at all, unless a *simple* substitution (see my manuscript on 7.1 under 'camouflage') handles them.

We also do not discuss a ∫impler case here, namely $\sqrt{ax+b}$: such a situation is naturally handled by the substitution $\sqrt{ax+b} = u$.

7.4.2: The three basic cases

$\boxed{\sqrt{a^2-x^2}:}$

You know that

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C\ .$$

This basic fact connects $\sqrt{a^2-x^2}$ under the integral with trig functions, namely with the sine: If you substitute $x = a\sin\theta$, then the integral will reduce to $\int d\theta = \theta + C$. (I assume $|\theta| \le \pi/2$ here, which makes $\sqrt{a^2-x^2} = a\cos\theta$ rather than $-a\cos\theta$.) Indeed, using the notation of differentials,

$$\tfrac{dx}{\sqrt{a^2-x^2}} = d\left(\arcsin \tfrac{x}{a}\right) = d\left(\arcsin \tfrac{a\sin\theta}{a}\right) = d\theta\ .$$

As a rule of thumb, you should learn from this example that, whenever you encounter a $\sqrt{a^2-x^2}$ under the integral sign somewhere, with nothing worse than rational expressions in x around otherwise, then you are likely to end up with an antiderivative that involves $\arcsin \frac{x}{a}$ somewhere.[1]

The explanation for this rule of thumb is based on that very trig substitution $x = a\sin\theta$, $\sqrt{a^2-x^2} = a\cos\theta$, $dx = a\cos\theta\, d\theta$. The substitution makes the integrand a rational expression in trig functions alone; examples can be found going from 7.4.3 to 7.4.4. In some cases, the integral can be done (not quite, but reasonably) easily, and if you end up with an isolated θ somewhere, back substitution turns it into $\arcsin \frac{x}{a}$.

You should now try to apply the technique to the following examples:

$$\int \sqrt{1-x^2}\, dx \qquad \text{and} \qquad \int (1-x^2)^{3/2}\, dx$$

More sophisticated examples can be handled by that very same substitution, but will then require, in the next step, a more sophisticated training to actually do the integrals that result from our trig substitution. See 7.4.4 below. Here, we are only concerned with the first step, namely choosing and using the trig substitution to transform an integral into a –hopefully– easier one.

$\boxed{\sqrt{x^2-a^2}:}$

In full analogy to the previous case, remembering the fact that[2]

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{ar\,cosh} \frac{x}{a} + C\ ,$$

[1] There may be exceptions, as in $\int (x/\sqrt{a^2-x^2})\, dx$, where the derivative of a^2-x^2, namely $-2x$, happens to hang around in the numerator– except for the constant factor -2, which does not change the structure of the expression. $\int (x/\sqrt{a^2-x^2})\, dx$ should be considered as structurally simpler than $\int (1/\sqrt{a^2-x^2})\, dx$. A trig substitution would still work very well on it, but would be overkill.

[2] Remember that, when I write ar cosh etc., I mean the same thing that the book calls $\cosh^{-1}$ etc.

you may deal with this square root by means of the substitution
$x = a\cosh t, \quad dx = a\sinh t\,dt, \quad \sqrt{x^2 - a^2} = a|\sinh t|$.

An alternative is (and this is the only choice the text book teaches you) to substitute
$x = a/\cos\theta = a\sec\theta, \quad dx = a(\sin\theta/\cos^2\theta)\,d\theta = a\sec\theta\tan\theta\,d\theta, \quad \sqrt{x^2 - a^2} = a|\tan\theta|$.
Here, the trig identity $\sec^2\theta - 1 = \tan^2\theta$ has been used. It arises from $\sin^2\theta + \cos^2\theta = 1$ after dividing by $\cos^2\theta$.

You may want to look up the formula $\int ??? = \operatorname{arcsec}\frac{x}{a} + C$ *in the book and compare the integrand with the type of square root under discussion here. You may also want to put all this info, i.e., derivatives of all inverse trig and hyp functions, in an orderly table. — No, I think I'd rather not prepare this table for you. If you do it yourself, it will help you much better to find a way through the inverse trig and hyp functions. — Part of this task is to organize the data in such a way that the arrangement of the formulas reflects similarities and analogies between them best possible.*

$\boxed{\sqrt{x^2 + a^2}:}$

Remembering the fact that

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{ar\,sinh}\frac{x}{a} + C\ ,$$

you may deal with this square root by means of the substitution $x = a\sinh t$, $dx = a\cosh t\,dt$, $\sqrt{x^2 + a^2} = a\cosh t$.

Here, the text book's choice is to substitute $x = a\tan\theta$, $dx = a\,d\theta/\cos^2\theta = a\sec^2\theta\,d\theta$, $\sqrt{x^2 + a^2} = a/\cos\theta = a\sec\theta$. The very same trig identity $1 + \tan^2\theta = \sec^2\theta$ has been used. *(Locate the relevant cases in your table of derivatives of inverse trig and inverse hyp functions.)*

How to remember these substitutions?

If you stick with the trig substitutions alone, you need to remember the trig identity that is responsible for making the term under the square root a complete square: namely, either $1 - \sin^2\theta = \cos^2\theta$, or $\sec^2\theta - 1 = \tan^2\theta$, or $1 + \tan^2\theta = \sec^2\theta$ respectively. They are all equivalent, and are all trig' versions of Pythagoras' theorem. I will not repeat here the figures of right triangles drawn in the textbook to illustrate the situation. Have a look at these figures there. Clearly, if you have a term $\sqrt{a^2 + x^2}$, this should be the hypotenuse of a right triangle, whereas for $\sqrt{a^2 - x^2}$, a should be the hypotenuse. Whichever type you have, θ will be one of the angles adjacent to the hypotenuse. It's not important which one. If your choice differs from the choice made in the textbook, you will get a somewhat different, but equally easy or equally difficult integral: in that case, switching between your solution and the one of the textbook will be done essentially by exchanging trig functions with their cofunctions: $\sin \leftrightarrow \cos$, $\tan \leftrightarrow \cot$, $\sec \leftrightarrow \csc$.

Another way to remember things is as follows: Whichever inverse trig or inverse hyp function has a derivative that involves a quadratic expression like the one under the square root, it will tell you an appropriate substitution: namely the corresponding direct (ie. not inverse) trig or hyp function. If you follow this advice and look up the derivative of the ar sech function (table 6.16 of the book), you will see that it involves a term $\sqrt{1 - x^2}$ (next to some other stuff). Therefore you *might* have used $x = a\,\operatorname{sech} t$ instead of $x = a\sin\theta$ to deal with a $\sqrt{a^2 - x^2}$, which was the very first case, where I did not put a hyp substitution next to the book's trig substitution. I could have given this alternative above already, so that all three types are treated on the same footing. I omitted it because I think it will rarely be convenient.

The options which I have given first in each case are those where the derivative of the inverse function is most similar to the square root under discussion: refer to your table of inverse trig and hyp functions to appreciate this similarity.

Ok, so you may ask yourself, because $\frac{d}{dx}\arcsin x = 1/\sqrt{1-x^2}$, I'll substitute $x = a\sin\theta$ to get rid of a square root $\sqrt{a^2-x^2}$, but not to get rid of $\sqrt{x^2-a^2}$. But how should I remember which inverse trigs / hyps have which derivative?

Well, the inverse sine and cosine, as well as their hyperbolic siblings, will have square roots in the derivative, whereas the inverse tangent and cotangent, as well as their hyperbolic siblings, don't. That's still manageable. But now, which was $\frac{d}{dx}\operatorname{ar\,sinh} x$? Was it $1/\sqrt{x^2+1}$ or was it $1/\sqrt{x^2-1}$ or was it $1/\sqrt{1-x^2}$? Or maybe the negative of any of these? — I suggest you remember the*graphs* of sin, cos, tan, and their hyperbolic siblings. Reflection gives you the graphs of their inverse functions immediately. Clearly arcsin is defined on $[-1,1]$, whereas ar sinh is defined for any number, and ar cosh is defined on $[1,\infty)$ only. So essentially the same should hold for their respective derivatives (we wouldn't quibble about the boundary points of these intervals). This information is enough to match the choices correctly.

Similar approach for tangent, cotangent, and their hyperbolic siblings. *You may enhance the table suggested at the top of page 3 with a few graphs.*

7.4.3: Some Examples

Try to handle the following examples with the trig and hyp substitutions offered above. Compare the results. (If you insist on not trespassing the borders of the textbook, do the trig subs only.) In a later step, we can discuss, how you would actually evaluate the integrals thus obtained.

(a) $$\int_0^2 \frac{\sqrt{x^2+4}}{x+2}\,dx$$

(b) $$\int_{-3}^3 \frac{\sqrt{9-x^2}}{9+x^2}\,dx$$

(c) $$\int_3^\infty \frac{\sqrt{x^2-9}}{(x^2+4)(x^2+1)}\,dx$$

(d) $$\int \frac{1}{(x^2+1)^2}\,dx$$

7.4.4: Rational Expressions in $\sin\theta$ and $\cos\theta$, and the $\tan\theta/2$ Substitution

Here are a few examples: actually, they are just the ones you got from using the appropriate trig substitutions in 7.4.3, and you may take this as a control for your results. Possibly, to make your results *look* the same, you need to kick out all the sec, tan etc. in favor of sin and cos only.

(a) $$\int_0^{\pi/4} \frac{2}{(\cos\theta+\sin\theta)\cos^2\theta}\,d\theta$$

$$(b) \qquad \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{1+\sin^2\theta}\,d\theta$$

$$(c) \qquad \int_0^{\pi/2} \frac{9\sin^2\theta\cos\theta}{(9+\cos^2\theta)(9+4\cos^2\theta)}\,d\theta$$

$$(d) \qquad \int \cos^2\theta\,d\theta$$

From your homework 7.4 #33, you know that all integrals of rational expressions involving only $\cos\theta$ and $\sin\theta$ can be transformed into integrals of rational functions by means of the substitution $y = \tan(\theta/2)$. I will comment on what is special about this substitution below. For the time being, you can use the examples to try that substitution. And you can consider whether what you get looks like progress as compared to the integrals with which you started in 7.4.3.

Certainly, if you have merely *polynomials* in $\sin\theta$ and $\cos\theta$, as in (d), the substitution $y = \tan\theta/2$ is the hard way to do it. Try to remember two essentially different alternative ways of evaluating (d).[3] I will also point out shortcuts for (b), (c) below. However, for (a), the substitution discussed here is probably *the* method of choice. $y = \tan\theta/2$ will *always* transform rational expressions of $\sin\theta$ and $\cos\theta$ into rational expressions of y, but in some cases there may be more efficient methods.

This is the point, where you should try this substitution on all four examples. And if you want to do it without being influenced by seeing the result, do it before reading ahead.

7.4.5: Assessing the $\tan\theta/2$ Substitution

The results which you get from 7.4.4 are:

$$(a) \qquad \int_0^{\tan\pi/8} \frac{4(1+y^2)^2}{(1-y^2)^2(1+2y-y^2)}\,dy$$

$$(b) \qquad \int_{-1}^{1} \frac{2(1-y^2)^2}{(1+y^2)((1+y^2)^2+4y^2)}\,dy$$

$$(c) \qquad \int_0^{1} \frac{72y^2(1-y^2)}{[9(1+y^2)^2+(1-y^2)^2][9(1+y^2)^2+4(1-y^2)^2]}\,dy$$

$$(d) \qquad \int \frac{2(1-y^2)^2}{(1+y^2)^3}\,dy$$

[3] Answer: either use trig identity $\cos^2\theta = (1+\cos 2\theta)/2$, or integrate by parts once, to get $\sin^2$ under the integral, then use trig' Pythagoras to get old integral back and solve for it.

Well, these are the results; (b) and (c) would still merit some algebraic simplification before doing a partial fractions decomposition. I will not suggest you should do these decompositions; however I hope you are aware by now, that, in principle, they could be done with the methods you have available, and with a lot of perseverance.

In cases (a)-(c), you *did* actually make progreſſ with these substitutions, because you got rid of the square root. In case (d), 7.4.4 was still progress, but now you are definitely worse. After a partial fraction decomposition, you would get $\int dy/(1+y^2)^2$, which is what you started with, and the worse integral $\int dy/(1+y^2)^3$ on top of it.

In case (a), you should probably accept the ugly result as something that reflects the actual difficulty of the problem. (The hyp substitution will work somewhat better.) In cases (b), (c), even though it *is* progreſſ to have killed the square root, you could have done better, as I will mention in 7.4.6.

$\tan\theta/2$ is like a 4WD jeep: it's not pretty and often not convenient, but it will run bravely even under adverse conditions like (a). The examples in the textbook are fine-tuned to be nicer than is common.

7.4.6: Why $\tan\theta/2$ is Natural, plus one Shortcut

The material in 7.4.6 goes beyond the textbook and is not required. You may want to read this leisurely now and look back at it as a reference later, but may not want to spend much effort on it for exam purposes.

Any expression $f(\theta)$ that is a rational expression in $\sin\theta$ and $\cos\theta$ alone has the property that $f(\theta+2\pi) = f(\theta)$, because the sine and cosine functions have this property. The natural geometric interpretation of the arguments of trig functions are *angles*. Irrespective of what the limits of integration in a particular example will be, you can think of θ as lying between $-\pi$ and π, because beyond this interval, the function values will just repeat. Now, if you look at $y = \tan\theta/2$, this substitution transforms an interval of length 2π (namely the interval $(-\pi,\pi)$) into the real line in a <u>one-to-one</u> way. Other substitutions you might consider as competitors (like, eg., $y = \sin\theta$, after all, there are sines in the integral, but no tangents!?) do not have this nice property:

To show why the competitor $u = \sin\theta$ couldn't work, I will use the word "different" in a special meaning focusing on the function f now (call it different): just for now, I will consider θ, $\theta+2\pi$, $\theta+4\pi$ etc. as NOT different, because, from the restricted point of view of the function f, they cannot be distinguished: $f(\theta+2\pi) = f(\theta)$. With $u = \sin\theta$, different numbers θ may be merged into a single u, like $\sin\frac{\pi}{4} = \sin\frac{3\pi}{4}$. Now $f(\frac{\pi}{4})$ may very well be different from $f(\frac{\pi}{4})$, but $u = \sin\theta$ will be the same for $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. So clearly, $f(\theta)$ couldn't be expressed in terms of u. This is why you couldn't tolerate your new variable u to merge different values of the old variable θ into a single value.

Another competitor, $u = \tan\theta$, i.e. without the half-angle, would suffer from the same deficit: $\tan(-\frac{\pi}{4}) = \tan(\frac{3\pi}{4})$, but $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$ have to be considered as different, because $f(-\frac{\pi}{4})$ may well be different from $f(\frac{3\pi}{4})$. After all, we only know $f(\theta) = f(\theta+2\pi)$, but couldn't expect $f(\theta) = f(\theta+\pi)$ for an expression $f(\theta)$ involving, e.g., $\sin\theta$.

However, if some function g is expressed in terms of $\sin\theta\cos\theta$, $\sin^2\theta$ and $\cos^2\theta$ alone (i.e., if sin and cos come always multiplied in pairs, but never single), then g has a different opinion of what numbers θ should be viewed as different (let's refer to g's understanding as "diggerent"). g would NOT consider θ and $\theta+\pi$ as diggerent, because g is expressed in terms of $\sin\theta\cos\theta$, $\sin^2\theta$ and $\cos^2\theta$ alone, and all of these expressions take the same value for θ and $\theta+\pi$. Now, although $u=\tan\theta$ may merge different numbers θ into the same u, it would not do the same thing to diggerent θ any more. In 7.4.4, the integrands (b) and (d) qualify as such functions g, but the integrands of (a) and (c) don't. And this is why the substitution $u=\tan\theta$ *is* a successful competitor in cases (b) and (d), but not in cases (a) and (c). If you tried it in these latter cases you would reintroduce square roots again.

You could have had a similar hwk problem like 7.4#33 for the $u=\tan\theta$ substitution instead of the $y=\tan\theta/2$ substitution, but we wouldn't make you spend the extra time for that one. If you use this substitution, you get the following:

$$(b)\qquad \int_{-\infty}^{\infty}\frac{1}{(1+u^2)(1+2u^2)}\,du$$

$$(d)\qquad \int\frac{1}{(1+u^2)^2}\,du$$

Clearly, they are much better than their siblings from 7.4.4. So, *if* your rational expression in $\sin\theta$ and $\cos\theta$ has the property that the trigs come in pairs, you will be better off with $u=\tan\theta$, but if they don't, then that choice isn't available.

Example (d) is still no progress, because we are just back at what we had in 7.4.3. No wonder, if we first substitute $x=\tan\theta$, then $\tan\theta=u$, what else would we get but $x=u$. But I had already promised you in 7.4.4 that these substitutions aren't helpful for something as easy as *polynomial* expressions in $\sin\theta$ and $\cos\theta$. If you believed me then, you won't be surprised now. Footnote 3 was the way to go in such $\int$impler cases.

Example (c) offers a special shortcut, if you see it. And the only thing that is around that could blindfold you into not seeing it is that we have been focusing on the tangent subs that long. Before this chapter, you would probably have found it less difficult to see than now. Can you still see the less sophisticated substitution?[4] It will lead you to the integral

$$(c)\qquad \int_0^1\frac{9w^2}{(10-w^2)(13-4w^2)}\,dw$$

So, (b) and (c) have become easy enough to serve as a manageable problem already.

7.4.7: And the Hyp Substitutions, What Happened to Them?

Well, you could try them on examples (a)–(d), starting from 7.4.3. As I don't love the secant function in general, I will omit the option $x=3\,\mathrm{sech}\,t$ for (b). The trig substitution *is* very natural here, whereas for (a), (c) the hyps are strong competitors. You may try them, and here are the results:

[4] Answer: If you write $\cos^2\theta = 1-\sin^2\theta$, you can substitute $w=\sin\theta$.

$$(a) \qquad \int_0^{\operatorname{ar\,sinh} 1} \frac{2\cosh^2 t}{1+\sinh t}\,dt \qquad \text{using } x = 2\sinh t$$

$$(c) \qquad \int_0^{\infty} \frac{9\sinh^2 t}{(9\cosh^2 t+4)(9\cosh^2 t+1)}\,dt \qquad \text{using } x = 3\cosh t$$

$$(d) \qquad \int \frac{1}{\cosh^3 t}\,dt \qquad \text{using } x = \sinh t$$

And just in case you wonder what to do with these integrals, by analogy, you may try either $z = \tanh t/2$ or, in (c), $v = \tanh t$. But here is another option: express all hyps in terms of exponentials, substitute $e^t = s$ and get a rational integrand in s. I will not carry out any of them here. And I would run out of variable names, if I explored more variants of substitutions.

7.4.8: An Afterthought – Take it Easy

I have packed very much material in this chapter. And I have introduced a world of ideas as motivation in 7.4.6 that will probably have been quite alien to the way you have been trained to think. Clearly you will not be required to master all of this material. The $\tan\theta/2$ substitution would have merited a chapter of its own, rather than being squeezed into a homework. You may be asked simpler problems using these substitutions. But I believe that you can master more easily what you are expected to master, if you have seen a larger picture. This detailed review is meant to help you organize your thoughts.

Moreover, if you start learning a foreign language, you may speak with a horrible accent in the beginning, you may use idioms inappropriately. You will be pardoned for this. But you wouldn't expect your teacher to speak with a horrible accent himself, or to use inappropriate idioms, just to do you a favor: because it wouldn't be a favor. I am following the same policy here, at a very crucial point of mathematical skill, namely handling integrals well. I know that this is a rather challenging thing for you and request your trust that it's worthwhile trying, and that the high-rising manuscript not meant to leave you behind. Deeper understanding will come step by step, when you reread the relevant sections, whenever a particular integral leaves you without a clue.

I wouldn't have invested many hours in this, but for the idea that you may find it helpful as a reference when you encounter all kinds of integrals later. It's intended beyond Math126.

You will often delegate the task of finding integrals to a computer. The results you will get from symbolic algebra software are correct most of the time; but they won't always be useful. Then your mathematical skills will be required again; the old-fashioned robust '4WD jeep type' of technology called ink and paper.

Good luck! And have a nice midterm break; you've merited it.