Appendix

from
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A. Geometry of Hypersurfaces.

In this section we list standard definitions and well-known geometric facts for hypersurfaces in $\mathbf{R}^{n+1}$. We consider only smooth embedded hypersurfaces M contained in some open set $U \subset \mathbf{R}^{n+1}$. Let the embedding map be denoted by $F : \Omega \to \mathbf{R}^{n+1}$ with $F(\Omega) = M$ where $\Omega \subset \mathbf{R}^n$ is open. M is called *properly embedded* if $F^{-1}(K) \subset \Omega$ is compact whenever $K \subset U$ is compact.

The coordinate tangent vectors $\partial_i F(p) \equiv \frac{\partial F}{\partial p_i}(p)$, $1 \le i \le n$ provide a basis of the *tangent space* $T_x M$ *at* $x = F(p)$ at every $p \in \Omega$. The *metric* on M is given by $g_{ij} = \partial_i F \cdot \partial_j F$ for $1 \le i, j \le n$, the *inverse metric* by $(g^{ij}) = (g_{ij})$ and the *area element* of M by $\sqrt{g} = \sqrt{\det g_{ij}}$. We are able to integrate compactly supported functions $h : M \to \mathbf{R}$ over a properly embedded hypersurface. The integral is defined by

$$\int_M h \equiv \int_M h \, d\mathcal{H}^n \equiv \int_M h(x) \, d\mathcal{H}^n(x) = \int_\Omega h(F(p)) \sqrt{g(p)} \, dp.$$

Here $\mathcal{H}^n$ denotes the n - dimensional Hausdorff measure on M (see section C). Note in particular that

$$\mathcal{H}^n(M \cap K) < \infty$$

for any compact $K \subset U$.

The *tangential gradient* of a function $h : M \to \mathbf{R}$ (which we may think of as a function on Ω via the embedding) is defined by

$$\nabla^M h = g^{ij} \partial_j h \, \partial_i F$$

where we sum over repeated indices. For a smooth tangent vectorfield $X = X^i \partial_i F = g^{ij} X_j \, \partial_i F$ on M (note that $X_i = X \cdot \partial_i F$) we define the *covariant derivative* tensor by

$$\nabla_i^M X^j = \partial_i X^j + \Gamma_{ik}^j X^k = g^{jl} \left(\partial_i X_l - \Gamma_{il}^k X_k \right)$$

where the *Christoffel symbols* Γ_{ij}^k are given by

$$(\partial_i \partial_j F)^T = \Gamma_{ij}^k \partial_k F$$

(here T denotes the tangential component of a vector) or in terms of the metric by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right).$$

The *tangential divergence of X on M* is defined by

$$\operatorname{div}_M X = \nabla_i^M X^i = g^{ij} \nabla_i^M X_j = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, g^{ij} X_j \right)$$

and the *Laplace-Beltrami operator* of h on M by

$$\Delta_M h = \operatorname{div}_M \nabla^M h = g^{ij} \left(\partial_i \partial_j h - \Gamma_{ij}^k \partial_k h \right) = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, g^{ij} \partial_j h \right).$$

For a smooth vectorfield $X : M \to \mathbf{R}^{n+1}$ which is not necessarily tangent to M we can also define the divergence with respect to M by

$$\operatorname{div}_M X = g^{ij} \partial_i X \cdot \partial_j F$$

which reduces to the above expression if X is tangent to M.

Let ν be a choice of *unit normal field* to M. In particular, this satisfies $\nu \cdot \partial_i F = 0$ on M for $1 \leq i \leq n$ and since ν has unit length, $\partial_i \nu$ is a tangent vector field to M for $1 \leq i \leq n$.

The *second fundamental form* of M is defined by

$$A_{ij} = \partial_i \nu \cdot \partial_j F = -\nu \cdot \partial_i \partial_j F.$$

The eigenvalues $\kappa_1, \ldots, \kappa_n$ of the *Weingarten map* (which is a map from the tangent space to itself) given by $A_j^i = g^{ik} A_{kj}$ are called the *principal curvatures of M*. The *mean curvature* can then be expressed in various forms by

$$H = \sum_{i=1}^{n} \kappa_i = A_i^i = g^{ij} A_{ij} = g^{ij} \partial_i \nu \cdot \partial_j F = \operatorname{div}_M \nu.$$

The *mean curvature vector of M* is given by $\vec{H} = -H\nu$. One checks the identity

(A.1) $$\Delta_M F = \vec{H}.$$ (see opposite page)

In the special case where $M = \operatorname{graph} u$, $u : \Omega \to \mathbf{R}$ (so $F(p) = (p, u(p))$, the upward unit normal vector ν is given by

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}$$

and the mean curvature of M by

$$-H = \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right)$$

where D and div denote the gradient and the divergence on $\mathbf{R}^n$.

The *Riemann curvature tensor* of M is defined by

$$\nabla_i^M \nabla_j^M X_k - \nabla_j^M \nabla_i^M X_k = R_{ijkl}^M X^l$$

where X is a tangent vectorfield on M. The *Gauss equations* express this tensor in terms of the second fundamental form by

$$R_{ijkl}^M = A_{ik}A_{jl} - A_{il}A_{jk}.$$

The *Codazzi equations* state that the 3-tensor of covariant derivatives of the second fundamental form $\nabla^M A = (\nabla_i^M A_{jk})$ is totally symmetric.

One defines covariant derivatives and the Laplacian of tensors analogously as in the case of vectorfields. Interchanging second derivatives for the second fundamental form and using the Gauss and Codazzi equations leads to Simons' identity

$$\Delta_M A_{ij} = \nabla_i^M \nabla_j^M H + H A_{ik} A_j^k - |A|^2 A_{ij} \tag{A.2}$$

(see [S] for the details of this computation). Here $|A|^2 = A_j^i A_i^j = g^{ij} g^{kl} A_{ik} A_{jl}$. Contracting this identity with $A^{ij} = g^{ik} g^{jl} A_{kl}$ yields the Bochner type formula

$$\Delta_M |A|^2 = 2\, A^{ij} \nabla_i^M \nabla_j^M H + 2\, |\nabla^M A|^2 + 2\, H A_{ij} A_{ik} A_{kj} - 2\, |A|^4 \tag{A.3}$$

where $|\nabla^M A|^2$ denotes the squared norm of the tensor $\nabla^M A = (\nabla_k^M A_{ij})$.

A calculation involving the Codazzi equations also implies the important identity

$$\Delta_M \nu = -|A|^2 \nu + \nabla^M H \tag{A.4}$$

(see for example [S] or [EH1]) which in integrated form yields the second variation formula for hypersurfaces.

Derivatives of functions and vector fields on hypersurfaces can also be defined in terms of projections from $\mathbf{R}^{n+1}$ to the tangent space of M. This is the framework used in geometric measure theory where coordinate systems are not available. The notions defined below carry over to n-rectifiable subsets of $\mathbf{R}^{n+1}$:

For $x \in M$ we define the projection $p_{T_x M} : \mathbf{R}^{n+1} \to T_x M$ by

$$p_{T_x M}(w) = w - (\nu(x) \cdot w)\nu(x).$$

Let $f : U \to \mathbf{R}$ be differentiable where U is an open subset of $\mathbf{R}^{n+1}$ containing M. We could alternatively also consider $f : M \to \mathbf{R}$ and require that is has a differentiable extension into an open neighbourhood of M in $\mathbf{R}^{n+1}$ (recall that derivatives of f in the direction of tangent vectors to M are independent of the particular extension of f). We define the *tangential gradient* of f with respect to M by

$$\nabla^M f(x) = p_{T_x M}(Df(x)) = Df(x) - \nu(x) \cdot Df(x)\, \nu(x)$$

for $x \in M$ where $Df(x)$ denotes the usual gradient of f (or of its extension into $\mathbf{R}^{n+1}$) in $\mathbf{R}^{n+1}$. This can also be written as

$$\nabla^M f(x) = \sum_{i=1}^{n} D_{\tau_i} f(x) \tau_i$$

where $D_{\tau_i} f(x)$ denotes the directional derivative with respect to τ_i and $\tau_1, \ldots, \tau_n$ form an orthonormal basis of $T_x M$.

For a differentiable vectorfield $X : U \to \mathbf{R}^{n+1}$ (or $X : M \to \mathbf{R}^{n+1}$) the derivative in the direction $w \in \mathbf{R}^{n+1}$ is given by

$$D_w X(x) = \left(\frac{\partial X_i}{\partial x_j}(x) \right)_{1 \le i,j \le n+1} \begin{pmatrix} w_1 \\ \vdots \\ w_{n+1} \end{pmatrix}.$$

The *tangential divergence of* X *with respect to* M is then defined by

$$\text{div}_M X(x) = \text{div}_{\mathbf{R}^{n+1}} X(x) - \nu(x) \cdot D_{\nu(x)} X(x). \tag{A.5}$$

Alternatively,

$$\text{div}_M X(x) = \sum_{i=1}^{n} \tau_i \cdot D_{\tau_i} X(x) = \text{trace}\left(p_{T_x M} \cdot DX(x)\right)$$

which is equivalent to the intrinsic formulation given above. The dot is used both for the dot product of vectors and composition of linear maps. It is usually clear from the context which one is meant. The *Laplace-Beltrami operator* on M of a twice differentiable function f is defined by

$$\Delta_M f = \text{div}_M \nabla^M f.$$

From the identity

$$\vec{H} = -H\nu = -(\text{div}_M \nu)\nu$$

we calculate using (A.5)

$$\begin{aligned} \Delta_M f &= \text{div}_M Df - \text{div}_M((\nu \cdot Df)\nu) = \text{div}_M Df - H\nu \cdot Df \\ &= \text{div}_M Df + \vec{H} \cdot Df = \Delta_{\mathbf{R}^{n+1}} f - D^2 f(\nu, \nu) + \vec{H} \cdot Df \end{aligned} \tag{A.6}$$

where $D^2 f(\nu, \nu) \equiv \nu \cdot D_\nu Df$ is the second derivative of f in normal direction to M.

If $f(x) = x_i$ for $1 \leq i \leq n+1$ then one calculates from (A.6) that $\Delta_M x_i = \vec{H} \cdot e_i$ (e_i is the i^{th} basis vector in $\mathbf{R}^{n+1}$) which implies the identity

$$\Delta_M x = \vec{H}$$

stated as (A.1) above.

The *Divergence Theorem* for smooth, properly embedded hypersurfaces states that for any C^1-vectorfield $X : M \to \mathbf{R}^{n+1}$ with compact support

$$(A.7) \qquad \int_M \mathrm{div}_M X = -\int_M \vec{H} \cdot X.$$

For a function $\phi \in C_0^2(\mathbf{R}^{n+1})$ the divergence theorem implies

$$(A.8) \qquad \int_M \mathrm{div}_M D\phi = -\int_M \vec{H} \cdot D\phi$$

or equivalently

$$(A.9) \qquad \int_M \Delta_M \phi = 0$$

in view of (A.6). For $\phi \in C_0^2(\mathbf{R}^{n+1})$ and $\eta \in C^2(\mathbf{R}^{n+1})$ one also checks that

$$(A.10) \qquad \int_M \phi \, \Delta_M \eta = -\int_M \nabla^M \phi \cdot \nabla^M \eta = \int_M \eta \, \Delta_M \phi.$$

B. Evolution Equations for Mean Curvature Flow.

In this section we recall the derivation of the evolution equations for most of the geometric quantities used in chapter 2. These were first derived in [Hu1], see also [GH] for the case of curves.

We will only have to calculate the time derivative of these quantities and then combine this with the identities for the Laplacian stated in section A. For ease of notation during calculations we will denote time derivatives by ∂_t.

We start with expressions involving the metric and calculate

$$(B.1) \qquad \partial_t g_{ij} = 2\, \partial_t \partial_i F \cdot \partial_j F = 2\, \partial_i \partial_t F \cdot \partial_j F$$

where we have used the fact that coordinate derivatives of $F(p,t)$ commute. Since $\partial_t F = -H\nu$ this becomes

$$\partial_t g_{ij} = 2\, \partial_i(-H\nu) \cdot \partial_j F = -2\, H \partial_i \nu \cdot \partial_j F$$

where we used $\nu \cdot \partial_j F = 0$ for $1 \leq j \leq n$. This implies

$$(B.2) \qquad \partial_t g_{ij} = -2\,H A_{ij}.$$

One easily checks that the inverse metric and the area element then satisfy

$$(B.3) \qquad \partial_t g^{ij} = 2\,H A^{ij}$$

and

$$(B.4) \qquad \partial_t \sqrt{g} = -H^2 \sqrt{g} = -|\vec{H}|^2 \sqrt{g}$$

since the derivative of g is given by $\partial_t g = g\, g^{ij} \partial_t g_{ij}$. Note that (B.1) also implies the more general formula

$$(B.5) \qquad \partial_t \sqrt{g} = \sqrt{g}\, g^{ij} \partial_i \partial_t F \cdot \partial_j F = \sqrt{g}\, \mathrm{div}_{M_t} \frac{\partial F}{\partial t}.$$

We continue with the time derivative of the second fundamental form. For convenience, we will calculate in geodesic normal coordinates on M_t that is assume $(\partial_i \partial_j F(p,t))^T = 0$ (T stands for the tangential component) at the point $x = F(p,t) \in M_t$ where we do the calculation. Since $A_{ij} = -\partial_i \partial_j F \cdot \nu$ we compute at $x = F(p,t)$

$$\begin{aligned} \partial_t A_{ij} &= -\partial_t (\partial_i \partial_j F \cdot \nu) \\ &= -\partial_i \partial_j \partial_t F \cdot \nu - \partial_i \partial_j F \cdot \partial_t \nu \\ &= -\partial_i \partial_j \partial_t F \cdot \nu \end{aligned}$$

since $\partial_t \nu$ is tangent and $\partial_i \partial_j F$ has no tangential component. We substitute $\partial_t F = -H\nu$ and again use $\partial_j \nu \cdot \nu = 0$ to obtain

$$\partial_t A_{ij} = \partial_i \partial_j H + H \partial_i \partial_j \nu \cdot \nu$$

and therefore

$$\partial_t A_{ij} = \partial_i \partial_j H - H \partial_i \nu \cdot \partial_j \nu.$$

Since in normal coordinates $\nabla_i^{M_t} \nabla_j^{M_t} H = \partial_i \partial_j H$ we conclude that

$$(B.6) \qquad \partial_t A_{ij} = \nabla_i^{M_t} \nabla_j^{M_t} H - H A_{ik} A_j^k.$$

The identities (B.6), (B.3) and $H = g^{ij} A_{ij}$, therefore yield

$$(B.7) \qquad (\partial_t - \Delta_{M_t})\, H = H|A|^2.$$

Moreover, one easily derives the evolution equation for A^{ij} and then checks that

$$(B.8) \qquad \partial_t |A|^2 = 2\, A^{ij} \nabla_i^{M_t} \nabla_j^{M_t} H + 2\, H A_{ik} A_j^k A^{ij}$$